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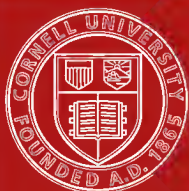
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ELEMENTS
OF THE
DIFFERENTIAL AND INTEGRAL
CALCULUS

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ELEMENTS
OF THE
DIFFERENTIAL AND INTEGRAL
CALCULUS

By

A. E. H. LOVE, M.A., D.Sc., F.R.S.

Sedleian Professor of Natural Philosophy in the University of
Oxford. Honorary Fellow of Queen's College, Oxford.
Formerly Fellow and Lecturer of St John's College, Cambridge

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PREFACE

IN the last six years I have given annually a course of about twenty lectures on the Elements of the Differential and Integral Calculus to classes consisting chiefly of students of Chemistry and Engineering. The work of preparing and delivering such lectures, and of revising them from year to year, teaches the lecturer many things in regard to the nature of the difficulties which are encountered by students. He is led to depart frequently from the traditional order of the subject matter, and to devise numerous simplifications in the proofs of propositions. It soon appeared that the amount of mathematical knowledge which need be possessed by a student before attempting the Calculus is very much less than has been supposed. For example, the Binomial Theorem in Algebra and the Addition Equation in Trigonometry are quite unnecessary. This book is written with the view of making the subject more easily and generally accessible than it has been hitherto. The principles of the Differential and Integral Calculus ought to be counted as a part of the intellectual heritage of every educated man or woman in the twentieth century, no less than the Copernican system or the Darwinian theory. In order to make a beginning no previous knowledge of mathematics is needed beyond the most elementary notions of geometry, a little algebra,

including the law of indices, and the definitions of the trigonometric functions. In order to advance very far in the subject a student must advance in other branches of mathematics as well. This book is intended merely to help the reader to make a beginning. In order to render his progress as easy as possible, results with which he is supposed to be more or less familiar are recapitulated in the places where they are wanted, and formal proofs of some propositions are omitted from the text and placed in Appendices, along with certain rather abstract discussions.

Two things in the subject are, and apparently must continue to be, difficult. These are the actual integration of particular functions and the theory of the exponential function. For a reader in search of culture the practice of integration is not very important, but for a student who wishes to make use of the Calculus it is indispensable. The difficulty seems to be purely one of *technique*. The best that I can do to meet it is to lead up gradually to the appropriate methods, and to illustrate them by sufficiently numerous and sufficiently easy examples. The student must not allow himself to be discouraged too easily by a few failures. The theory of the exponential function, on the other hand, is essentially difficult, and the history of mathematics shows what a formidable stumbling-block it proved itself from the time of the invention of logarithms by John Napier to the time of the revision of the foundations of mathematical analysis by Augustin Louis Cauchy. For the purpose of a study of the elements of the Calculus the whole of the theory is not required, for example, the exponential theorem is unnecessary, but it is necessary either to prove or to assume at least one

proposition the rigorous proof of which is difficult. My plan has been to assume the existence of the exponential limit. This limit presents itself naturally in the process of differentiating a logarithm, while logarithms arise naturally, though not historically, from the use of indices of powers. It would be unsatisfactory to assume the existence of the limit without explanation, and for this reason an arithmetical argument has been given which makes it appear probable that the limit exists. Such arguments are often more convincing than formal proofs. It would be unsatisfactory to substitute such an argument for a proof, and merely irritating to refer the reader to a proof in some book which he may not possess, and a formal proof is given in an Appendix (pp. 191—194). I have not tried to select one which is brief, or easy to reproduce in examinations, but my choice was guided by the wish to use none but the simplest mathematical material.

The student who wishes to master the subject within the range of this book is recommended to work the Examples. In working some of these, a book of Tables of mathematical functions is needed, and it is assumed that the reader knows how to compute by means of logarithms. This knowledge is not, however, required in order to read the greater part of the text.

My best thanks are due to Mr F. B. Pidduck, Fellow of Queen's College, Oxford, for his kindness in reading and correcting the proofs.

A. E. H. LOVE.

OXFORD,
July 1909.

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CORRIGENDA

- Page 15, line 14, *for* " $\frac{1}{3}$, (0·1)" *read* " $\frac{1}{3}$ (0·1)."
- „ line 15, *for* "0·95561" *read* "0·95661."
- „ line 18, *for* "Ex. 15" *read* "Ex. 16."
- „ 93, ftn., *for* "1894" *read* "1904."

CHAPTER I

INTRODUCTORY

1. MANY quantities that we know how to measure are variable, for example, the temperature of the air, or the speed of a train. When a quantity can vary, it usually depends upon other quantities which can also vary. We shall consider some examples of relations between the measures of variable quantities.

(a) *Weighing with a spring balance.* Before any weight is hung on, the spring has a certain length. Let this length be b inches. Then b is a certain number; it may be not a whole number, but that does not matter. When a weight is hung on, the spring is stretched. When a weight of 1 lb. is hung on, let the spring be stretched m inches, so that its length becomes $b + m$ inches. The number m would not generally be a whole number. When a weight of x lbs is hung on, let the length of the spring become y inches. If the weight is not too great the amount by which the spring is stretched is proportional to the weight, or we have

$$y - b : m = x : 1,$$

and this is the same as

$$y = mx + b.$$

The numbers x and y need not be whole numbers. The numbers m and b do not depend upon x or y , but the number y depends upon the number x .

(b) *Comparison of thermometric scales.* Let C degrees on the Centigrade scale specify the same temperature as F degrees on the Fahrenheit scale. The diagram (Fig. 1) gives at once the equation

$$\frac{F - 32}{180} = \frac{C}{100}$$

or
$$F = \frac{9}{5}C + 32.$$

If for $C, F, \frac{9}{5}, 32$ we write x, y, m, b , the equation becomes

$$y = mx + b.$$

Both x and y can be negative numbers.

In these two examples we have used the same letters x, y, m, b in order to bring out the similarity of *form* of the relations between the variable numbers that occur. The letters always stand for numbers, but the quantities of which these numbers are the measures are different in the different examples.

2. Values assigned to two variable numbers, such as the x and y of the previous examples, can be shown graphically. We draw on paper two straight lines, which cut each other at right angles, one running from left to right, called the "axis of x ," the other running up and down, called the "axis of y ." The point where they meet is called the "origin." We choose a unit of length, a foot for instance, or a centimetre, or the distance between two of the ruled lines if we are using squared paper. If x is a positive number, we can find any number of points on the plane of the paper which are to the right of the axis of y , and at a distance x units of length from it. All these points lie on a straight line parallel to the axis of y . (One such line is dotted in Fig. 2.) In like manner, if y is a positive number, we can find any number of points on the plane of the paper which are above the axis of x , and at a distance of y units of length from it. All these points lie on a straight line parallel to



Fig. 1.

the axis of x . The two lines meet in a point, which is distant x units of length to the right of the axis of y , and y units of length above the axis of x . The numbers x and y are called the

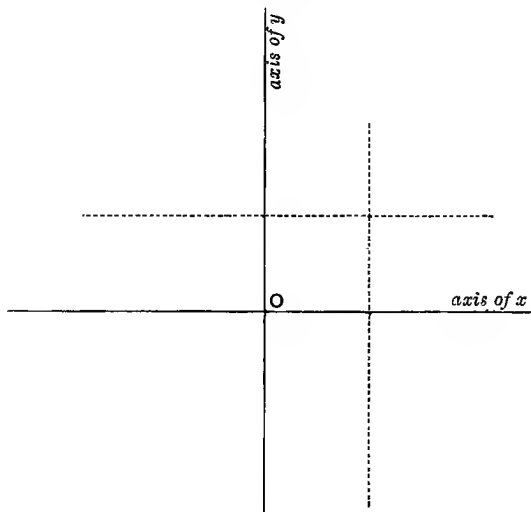


Fig. 2.

“coordinates” of the point. When x and y are chosen the point is fixed. We may say that the point “represents” the pair of numbers x and y .

When one, or both, of the numbers x , y is negative, we can still represent the pair by a point. If x is a negative number, $-x$ is a positive number, and we take the point, which represents the pair of numbers x and y , to be at a distance $-x$ units of length to the *left* of the axis of y . If y is a negative number we take the point to be at a distance $-y$ units of length *below* the axis of x . In this way we find one point, and only one, which represents a pair of numbers x , y . The point is often called the point (x, y) .

3. For example, in Fig. 3 the four points $(2, 3)$, $(2, -3)$, $(-2, 3)$, $(-2, -3)$ are marked with crosses, the distance between consecutive ruled lines being taken as the unit of length.

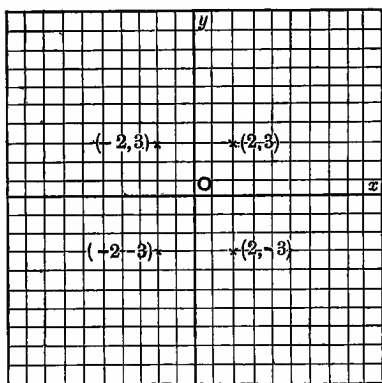


Fig. 3.

4. The axes of x and y divide the plane of the paper into four compartments or "quadrants." If from any point P we let

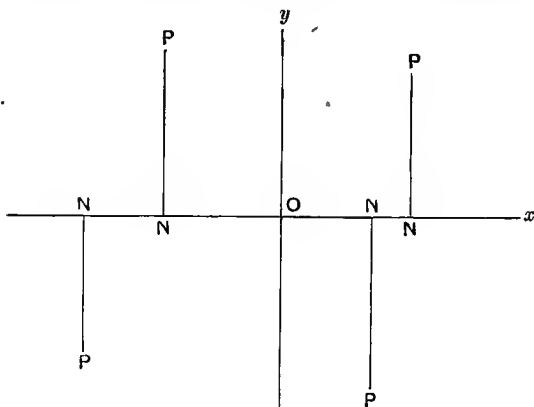


Fig. 4.

fall a perpendicular PN upon the axis of x , the straight line PN is called the "ordinate" of P and the straight line ON the "abscissa" of P (Fig. 4). The coordinates x, y , if we disregard their signs, are the measures of the lengths of the abscissa and ordinate. The coordinates, with their proper signs, tell us not only how long to make the abscissa and ordinate, but also in which of the four quadrants the point P lies.

5. We go back to the equation

$$y = \frac{9}{5}x + 32,$$

in which x and y are the two numbers which specify the same temperature on the Centigrade and Fahrenheit scales. We mark

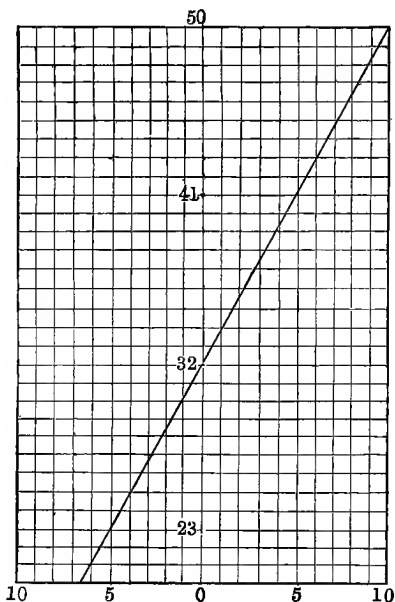


Fig. 5.

on squared paper some of the points whose coordinates satisfy

the equation. For instance some corresponding values of x and y are given by the table

x	-10	-5	0	5	10	15
y	14	23	32	41	50	59

We find that all the points which have these coordinates lie on a straight line. Part of this line is shown in Fig. 5. It is easy to prove formally that every pair of values of x and y by which the equation can be satisfied is represented by a point on this line, and that every point on this line represents a pair of numbers x, y which satisfy the equation. But we shall omit the proof for the present. The line is said to be the *graph* of the expression $\frac{9}{5}x + 32$ to which y is equal, or shortly "the graph of $y = \frac{9}{5}x + 32$." It is easy to prove formally that, if m and b are independent of x , the graph of $y = mx + b$ is a straight line*.

6. If a body, such as a stone, moves over a distance s feet in t seconds, its average velocity during this interval is $\frac{s}{t}$ feet per second. The body may move in such a way that this is the same for all values of t . It then moves "uniformly." If we

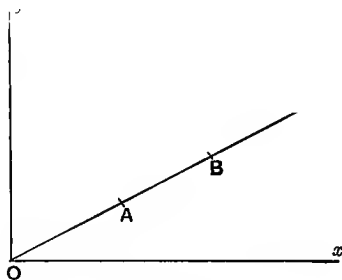


Fig. 6.

* A formal proof will be found in Appendix I.

write v for the constant value of $\frac{s}{t}$, the velocity of the body is v feet per second. If we write y for s and x for t , we have $y = vx$. The graph of this equation is the "distance-time" graph for the body.

Let A and B be two points of the graph, x_A, y_A the coordinates of A, x_B, y_B those of B. We take x_B to be greater than x_A as in Fig. 6. The number $x_B - x_A$ is the measure in seconds of a certain interval of time. The number $y_B - y_A$ is the measure in feet of the distance over which the body moves during that interval. The fraction

$$\frac{y_B - y_A}{x_B - x_A}$$

is the measure in feet per second of the average velocity in this interval. If, as above, $y = vx$, where v is independent of x , $y_B = vx_B$ and $y_A = vx_A$, so that the fraction is equal to v , and therefore has the same value for every interval.

7. If we have any straight line graph we can choose two points A and B on it, and form the fraction $\frac{y_B - y_A}{x_B - x_A}$, and this fraction has a simple graphic interpretation whether the straight line is a distance-time graph of a moving body or not.

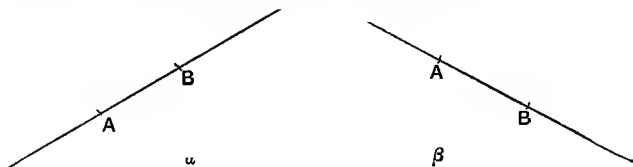


Fig. 7.

We take B to be further to the right than A, so that $x_B > x_A$. If B is higher up than A, $y_B > y_A$ (Fig. 7 α). If A is higher up than B, $y_B < y_A$ (Fig. 7 β).

In the first case the graph goes up to the right, and the fraction $\frac{y_B - y_A}{x_B - x_A}$ is positive. In the second case it goes down to

the right, and the fraction is negative. If we disregard the sign the fraction gives us a measure of the steepness of the graph. It is as if the straight line ran through the edges of a set of steps so that A and B are edges of two consecutive steps (Fig. 8). Then $x_B - x_A$ is the measure of the breadth of a step, and the absolute value of $y_B - y_A$ (sign disregarded) is the measure of the height of the step. The absolute value of the fraction $\frac{y_B - y_A}{x_B - x_A}$ (sign disregarded) tells us how steep the line is, and the sign of the fraction tells us whether the line goes up to the right or down to the right. The fraction with its proper sign is called the "gradient" of the line.

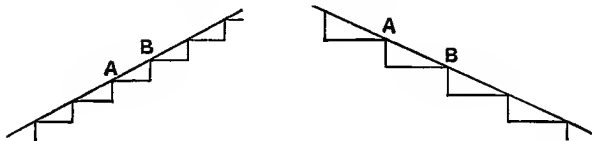


Fig. 8.

Since $\frac{y_A - y_B}{x_A - x_B} = \frac{y_B - y_A}{x_B - x_A}$, the gradient is the same whether B is to the right of A or to the left of A.

8. Let the straight line be the graph of $y = mx + b$. We have

$$y_B = mx_B + b, \quad y_A = mx_A + b,$$

so that

$$y_B - y_A = m(x_B - x_A),$$

and

$$\frac{y_B - y_A}{x_B - x_A} = m.$$

Hence the number m in the equation $y = mx + b$ is always the gradient of the corresponding graph. The graph goes up to the right or down to the right according as m is positive or negative.

If the straight line is the distance-time graph for a body moving uniformly, the gradient is the measure, in appropriate units, of the velocity of the body.

If the straight line is parallel to the axis of x , y is the same at all points of it. Then $y_B - y_A = 0$, and the gradient is 0.

EXAMPLES

Draw the graphs of the following, (1)–(17), and determine the gradient in each case :—

- (1) $y = x$, (2) $y = x + 1$, (3) $y = x - 1$, (4) $y = -x$,
 (5) $y = -x + 1$, (6) $y = -x - 1$, (7) $y = x + 2$, (8) $y = -x - 3$,
 (9) $y = 2x$, (10) $y = 2x + 4$, (11) $y = 2x - 1$, (12) $y = -2x$,
 (13) $y = -2x - 3$, (14) $y = \frac{1}{2}x$, (15) $y = \frac{1}{2}x - 1$, (16) $y = -\frac{1}{2}x$,
 (17) $y = -\frac{1}{2}x + 2$.

9. When a stone is let fall it begins to move very slowly, but after it has been moving for a little time it moves more quickly. The number of feet through which it falls in a few seconds is not always the same multiple of the number of seconds. In an interval of t seconds, reckoned from the instant when the stone is let fall, it moves a certain distance, which we take to be s feet. Neither t nor s need be a whole number. Now it is found that, apart from a correction depending on the resistance of the air,

$$s = (16 \cdot 1) t^2.$$

Put y for $\frac{s}{16 \cdot 1}$ and x for t . We get

$$y = x^2.$$

This equation is quite different from any that we have had before. We proceed to draw the corresponding graph. As before, we make a table, thus :—

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y	0	0.01	0.04	0.09	0.16	0.25	0.36	0.49	0.64	0.81	1

For convenience we take the distance between two consecutive ruled lines on the squared paper to be one-tenth of the unit of length. We mark the points which have the above coordinates and draw a smooth curve through them. This curve is part of

the graph of $y = x^2$. To complete the graph we should have to find the values of y that correspond to negative values of x , and

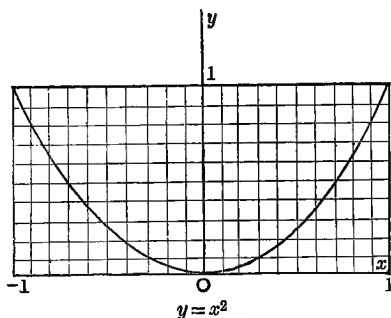


Fig. 9.

to values of x that are greater than 1. In Fig. 9 the graph is drawn for values of x that lie between -1 and 1 .

In this example the graph is a curve called a "parabola."

EXAMPLES

1. Plot the graph of $y = x^2$ between $x = -1$ and $x = 1$.
2. Plot the graph of $y = -x^2$ between $x = -1$ and $x = 1$.
3. Make a table of the values of $\frac{1}{10}x^3$ when x has the values $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$. Mark on squared paper the corresponding points of the graph of $y = \frac{1}{10}x^3$ and draw the graph.

10. In the case of the falling stone we had

$$s = (16 \cdot 1) t^2, \quad y = \frac{s}{16 \cdot 1}, \quad x = t, \quad y = x^2.$$

If we take $16 \cdot 1$ feet as the unit of length, the stone falls y units of length in x seconds, and the number y is the square of the number x . Let A, B be two points on the graph of $y = x^2$. In x_A seconds the stone falls y_A units of length, in x_B seconds it falls y_B units of length. We take x_B to be greater than x_A . Consider the interval of time which begins at the instant x_A

seconds after the instant when the stone was let fall, and ends at the instant x_B seconds after the instant when the stone was let fall. During this interval of $x_B - x_A$ seconds the stone falls $y_B - y_A$ units of length, and its average velocity is $\frac{y_B - y_A}{x_B - x_A}$ units of length per second.

For shortness write h for $x_B - x_A$, so that $x_B = x_A + h$. Then

$$y_A = x_A^2, \text{ and } y_B = x_B^2 = (x_A + h)^2 = x_A^2 + 2hx_A + h^2,$$

$$\text{and } \frac{y_B - y_A}{x_B - x_A} = \frac{x_A^2 + 2hx_A + h^2 - x_A^2}{h} = \frac{2hx_A + h^2}{h} = 2x_A + h.$$

During the interval of h seconds, beginning at the instant specified by x_A , the average velocity of the stone is $2x_A + h$ units of length per second. This average velocity depends not only on the value of x_A but also on the value of h .

If we keep x_A always the same, and take a smaller value for h , then a still smaller value, and so on, we see that we can bring the expression $2x_A + h$ as near to $2x_A$ as we please. We may express this by saying that, as h tends to zero, $2x_A + h$ tends to $2x_A$ as a *limit*.

Keeping x_A always the same is keeping the initial instant of an interval always the same. Diminishing h is shortening the interval. We have learnt that, as the interval is shortened, the initial instant remaining the same, the average velocity tends to a limiting velocity. When we speak of the velocity of the stone *at the instant* in question we mean this limiting velocity.

Further we have learnt that the velocity at the instant specified by x_A is $2x_A$ units of length per second. Our unit of length was 16.1 feet, and therefore the velocity is $(32.2)x_A$ feet per second. In other words, the velocity of the stone at the instant which is t seconds later than the instant at which it was let fall is $(32.2)t$ feet per second.

11. We consider the matter more generally by thinking of the graph of $y = x^2$ without regarding it as a distance-time graph. As before let the x -coordinates of two points A and B be x_A and $x_A + h$. Formerly we took B to the right of A, so that h was

positive, and we found that the gradient of the straight line AB was $2x_A + h$. But this result is unaltered if B is to the left of A, so that h is negative, for the algebraic work by which we found the result remains the same.

We found that, if h is positive, $2x_A + h$ tends to $2x_A$ as a limit when h tends to zero. If h is negative $2x_A + h$ is less than $2x_A$. If we take a value of h , negative but nearer to zero than before, then another nearer still, and so on, we see that we can bring $2x_A + h$ as nearly up to $2x_A$ as we please. Thus $2x_A + h$ tends to the same limit $2x_A$ whether h is positive or negative.

Bringing h nearer and nearer to zero is bringing B nearer and nearer to A. The gradient of the secant, or cutting line, AB is $2x_A + h$. It can be greater or less than $2x_A$, but cannot be equal to $2x_A$ for any secant drawn through A. As B is brought nearer and nearer to A, either on the right-hand side or on the left, the gradient tends to $2x_A$ as a limit. A straight line drawn through A, and having the gradient $2x_A$, would not cut the curve again, but if we increased or diminished the gradient ever so little it would cut the curve. The straight line in question touches the curve at A. It is the "tangent" to the curve at A (Fig. 10).

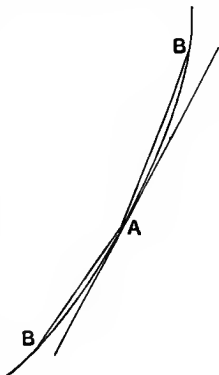


Fig. 10.

The number $2x_A$ is the gradient of the tangent to the curve at A. We may call it the "gradient of the curve" at A. We have learnt that the gradient of the parabola, given by $y = x^2$, at the point whose coordinates are x and x^2 , is $2x$.

12. The process here described is of very general application. It may be summed up in the rule:—Let A, B be two points of a curve. Find the gradient of the straight line AB. The limit to which this gradient tends, as B is brought nearer and nearer to A, is the gradient of the curve at A.

The business of the Differential Calculus is to determine in each case the limit to which the gradient of the secant tends. When we determine such a limit we *differentiate*. In the case of the parabola we differentiated x^2 . The result was $2x$.

In the case of a straight line graph we do not need to bring B nearer to A in order to determine the gradient. The gradient is 0 for the graph of $y = a$, it is m for the graph of $y = mx + b$. We may say that, when we differentiate a , the result is 0, and when we differentiate $mx + b$, the result is m , the numbers a , b , m being independent of x .

EXAMPLES

1. A moving body passes over s feet in t seconds, and s and t are connected by the equation $s = t^2$. What is the velocity of the body at the end of the fifth second? What is the average velocity of the body during the fifth second? [Results : 10 ft. per sec., 9 ft. per sec.]

2. As in Ex. 1, a body moves according to the law $s = t^2$. Find its average velocity in the first tenth, first hundredth, first thousandth of a second, beginning at the end of the fifth second.

3. The point $(1, 1)$ on the parabola, given by the equation $y = x^2$, is joined to the point $\{1 + h, (1 + h)^2\}$. Find the gradient of the secant when h has the values 0.1, 0.01, 0.001, and -0.1, -0.01, -0.001.

ADDITIONAL EXERCISES

Some additional examples of graphs and computing are placed here. These should not all be worked out before reading the rest of the book. The student will find that it is better to do one or two of these exercises every day.

1. Plot the graph of $y = \log_{10} x$ from $x = 0.1$ to $x = 10$ by giving to x the values 0.1, 0.2, ... 1 and 1, 2, ... 10.

2. Plot the graph of $y = 10^x$ from $x = -1$ to $x = 1$ by giving to x the values -1, -0.9, -0.8, ... -0.1, 0, 0.1, 0.2, ... 0.9, 1.

3. Make a table of the values of $x \log_{10} \left(1 + \frac{1}{x}\right)$ by giving to x the values 1, 2, 3, ... 9. Proceed to make a table of the values of $\left(1 + \frac{1}{x}\right)^x$ for the same values of x , and plot the graph of $y = \left(1 + \frac{1}{x}\right)^x$ between $x = 1$ and $x = 9$.

4. Do the same as in Ex. 3 with the values 10, 20, 30, ... 100 for x .

5. Plot the graph of $y = \frac{1}{x}$ by giving to x the values 0.1, 0.2, ... 0.9, 1, 2, and the same values with minus signs (-0.1, ...).

6. Draw on one piece of squared paper the graphs of $y = \frac{1}{10} x^3$ and $y = \frac{1}{10} (7x + 2)$. Find the x -coordinates of the points where they meet. [Results: 2.8, -0.4, -2.4 approximately.]

Note that this method gives the approximate solution of a cubic equation $x^3 - 7x - 2 = 0$.

7. Solve approximately the cubic equations

$$x^3 - 9x + 7 = 0 \quad \text{and} \quad 3x^3 - 5x + 1 = 0.$$

8. Plot the graph of $y = x + \frac{1}{x}$ from $x = 0.5$ to $x = 1.5$ by giving to x the values 0.5, 0.6, 0.7, ... 1.5.

9. Plot the graph of $y = x(1 - 2x)^2$ from $x = 0$ to $x = 1$ by giving to x the values $\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12}, 1$.

10. Plot the graph of $y = x(13 - 2x)(15 - 2x)$ from $x = 0$ to $x = 10$ by giving to x the values 0.5, 1, 1.5, 2, ... 9.5, 10.

11. Plot the graph of $y = x(13 - 2x)(15 - 2x)$ from $x = 6.5$ to $x = 7.5$ more minutely than in Ex. 10, by giving to x the values 6.5, 6.6, 6.7, 6.8, ... 7.4, 7.5.

12. Given $\log_{10} e = 0.4343$, make a table of the values of e^x for $x = 0$, 0.1, 0.2, 0.3, ... 0.9, 1.

13. Do the same for e^{-x} .

14. Plot a graph of $y = e^x$ from $x = -1$ to $x = 1$.

15. Plot a graph of $y = 1 - e^{-x}$ by taking $x = 0$, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5.

16. Given $y = \sqrt{1 - x^2}$, calculate, to six places of decimals, the values of y which correspond to the values -0.5 , -0.4 , -0.3 , -0.2 , -0.1 , 0, 0.1, 0.2, 0.3, 0.4, 0.5 of x . Denoting these values by y_1, y_2, \dots, y_{11} , find the value to five places of decimals of

$$\frac{1}{3}, (0.1) \{y_1 + y_{11} + 2(y_3 + y_6 + y_7 + y_9) + 4(y_2 + y_4 + y_6 + y_8 + y_{10})\}.$$

[Result: 0.95561.]

17. Given $y = \frac{1}{x}$, calculate, to six places of decimals, the values of y which correspond to the values 1, 1.1, 1.2, ... 1.9, 2 of x . Proceed as in Ex. 15. [Result: 0.69315.]

We shall use the Results of Exs. 16 and 17 in Ch. X.

CHAPTER II

DIFFERENTIATION

13. WE have considered a few examples in which one variable number y is expressed in terms of another variable number x by an equation. These numbers *may* be thought of as the measures of certain variable quantities in terms of appropriate units, but they *need not*. The expression by means of which the value of y can be written down when a value of x is chosen may be much more complicated than any of those which we considered. But, whether the expression is simple or complicated, we think of it as equal to another number y , which is variable when x is variable. When we think of an expression containing x as being itself a variable number, which takes various values according to the value given to x , we call it a "function" of x . For instance x^2 , $\log_{10} x$, $\sin x$ are functions of x . We use the notations $f(x)$, $F(x)$, $\phi(x)$ &c. to denote functions of x . It is important not to think of the symbol f in $f(x)$ as a number, but to regard the expression " $f(x)$ " as an abbreviation for "a function of x ." When we draw, or plot, a graph by using an equation of the form $y=f(x)$, we draw, or plot, the "graph of the function." All the functions that we shall have to consider possess graphs*.

14. We found that we could determine the gradient of the graph at a point by a certain process whenever we could carry

* For a limitation implied in this statement see Appendix II.

out the process. The process was this:—Let x_A be the x of the point. In the expression $f(x)$, to which y is equal, substitute x_A for x . In the same expression substitute $x_A + h$ for x . Form the difference $f(x_A + h) - f(x_A)$, and divide it by h . Determine the limit to which the quotient

$$\frac{f(x_A + h) - f(x_A)}{h}$$

tends as h is diminished towards zero, if it is positive, or as h is increased towards zero, if it is negative. In the result drop the suffix A . Then we have the gradient of the graph at any point (x, y) on it.

We may omit the suffix A throughout the process, but when we do this we must keep it in mind that throughout the process x is a constant number and h is the only variable.

15. We shall now write Δx instead of h . The symbol Δ (Delta) is not to be thought of as a number multiplying x , but the expression Δx is to be thought of as itself denoting a number. It means “a number added to x .” This number Δx may be positive or it may be negative. When we replace x in $f(x)$ by $x + h$ or $x + \Delta x$, obtaining the expression $f(x + \Delta x)$, we may use a number Δy to stand for the difference $f(x + \Delta x) - f(x)$, so that

$$y + \Delta y = f(x + \Delta x),$$

y being the number to which $f(x)$ is equal; and then Δy is “the number that is added to y when Δx is added to x .”

The quotient

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is the same as

$$\frac{\Delta y}{\Delta x},$$

and we have to determine the limit to which this quotient tends when Δx tends to zero. We write

$$\frac{dy}{dx}$$

for this limit. This symbol is not to be thought of as a fraction or quotient, but as the limit to which the quotient $\frac{\Delta y}{\Delta x}$ tends as Δx tends to zero. We need not attempt to give a meaning to dy or to dx .

The limit expressed by the symbol $\frac{dy}{dx}$ is called the "differential coefficient of y with respect to x ."

16. When the limit is found it appears that it depends upon the value assigned to x . After finding it we may again regard x as a variable. For instance we found that when $y = x^2$, $\frac{dy}{dx} = 2x$, and $2x$ is a function of x . When we wish to think of the differential coefficient of $f(x)$ as a function of x we denote it by $f'(x)$. The function $f'(x)$ is often called the "derived function" of $f(x)$.

17. The results which we have found so far may be written in the notation of differential coefficients as follows:—

(i) If a is independent of x , $\frac{da}{dx} = 0$,

(ii) If m and b are independent of x , $\frac{d(mx + b)}{dx} = m$,

(iii) $\frac{d(x^2)}{dx} = 2x$.

In the result (ii) is included the result $\frac{dx}{dx} = 1$, which is obtained by putting $m = 1$, $b = 0$. This result has been proved by finding the gradient of the graph of $y = x$. It may not be inferred from the form of the symbol $\frac{dx}{dx}$ by thinking of this symbol as a fraction of which the denominator is equal to the numerator, because the symbol is not a fraction with dx for numerator and dx for denominator.

18. We know that one use of differential coefficients is to find the gradient of a graph. We now illustrate some further uses of them.

(a) *Falling body.* We consider a stone let fall. We know that, apart from a correction depending upon the resistance of the

air, the distance s feet through which it falls in t seconds is given by the equation

$$s = (16 \cdot 1) t^2.$$

In $t + \Delta t$ seconds it falls through $s + \Delta s$ feet, and we find

$$\Delta s = (16 \cdot 1) (2t + \Delta t) \Delta t.$$

During the particular interval denoted by Δt seconds it falls through Δs feet, and its average velocity in this interval is $\frac{\Delta s}{\Delta t}$ feet

per second. As Δt tends to zero the number $\frac{\Delta s}{\Delta t}$ tends to a limit which is $\frac{ds}{dt}$. We find

$$\frac{ds}{dt} = (32 \cdot 2) t.$$

The velocity $(32 \cdot 2) t$ feet per second is the velocity of the stone at the instant specified by t , that is to say t seconds after it was let fall.

(b) *Speed of moving body.* More generally, if a body is in motion it moves a distance s feet in t seconds, and $s + \Delta s$ feet in $t + \Delta t$ seconds. The number $\frac{\Delta s}{\Delta t}$ is the measure in feet per second of its average velocity in the interval denoted by Δt seconds, and the limit $\frac{ds}{dt}$, to which this number tends as Δt tends to zero, is the measure in feet per second of its velocity at the instant specified by t .

(c) *Rate of change in general.* Still more generally, if q is a number, which is the measure of a variable quantity in terms of some unit, and the instant at which the quantity is measured by q is t seconds later than some chosen instant, q is a function of t , and the differential coefficient $\frac{dq}{dt}$ measures the rate per second at which the quantity measured by q is increasing.

For example, in a vessel containing some water and some ice, q may be the number of lbs. of ice at the instant specified by t . If $\frac{dq}{dt}$ is positive,

it measures in lbs. per second the rate at which water is being converted into ice, or the rate of freezing. If $\frac{dq}{dt}$ is negative, $-\frac{dq}{dt}$ measures in lbs. per second the rate at which ice is being converted into water, or the rate of thawing.

(d) *Tangent to a curve.* We have already seen that $\frac{dy}{dx}$ is the gradient of the tangent to a graph at the point (x, y) . Let the tangent at a point P above the axis of x meet this axis in T, let PN be the ordinate of P, and suppose that the curve goes up to the right (Fig. 11). The fraction
$$\frac{\text{number of units of length in the length of NP}}{\text{number of units of length in the length of NT}}$$
 is the gradient of the tangent at P. A similar result holds in any case if we give the right sign to the fraction. (See Ch. IX.) It appears that when the gradient is given we can draw the tangent.

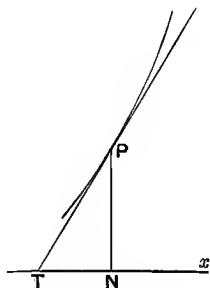


Fig. 11.

For example, take the parabola given by the equation $y = x^2$. We have $\frac{dy}{dx} = 2x$. Now PN contains x^2 units of length when ON contains x units

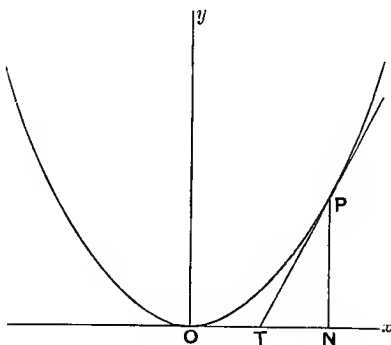


Fig. 12.

of length. Let TN contain z units of length. Then $\frac{x^2}{z} = 2x$, or $z = \frac{1}{2}x$, and therefore TN is half of ON . To construct the tangent at P , draw the ordinate PN , bisect the abscissa ON in T , and join TP . The straight line TP is the tangent to the parabola at the point P (Fig. 12).

EXAMPLES

1. A body is moving in such a way that the distance s feet passed over in any time t seconds from the start is proportional to the square of the velocity at the instant specified by t . Express this fact by an equation connecting $\frac{ds}{dt}$ with s .

2. A body is moving in such a way that its velocity, v feet per second at the instant specified by t , is increasing at a uniform rate. Express this fact by an equation containing $\frac{dv}{dt}$.

3. A body is moving in such a way that its velocity, v feet per second at the instant specified by t , is diminishing at a rate proportional to v . Express this fact by an equation.

4. A tank is being emptied in such a way that the rate at which the water flows out is proportional at any instant to the amount of water left in the tank at that instant. Express this fact by an equation.

5. One substance is being transformed into another according to the law that the rate of transformation per second is proportional at any instant to the amount that has not been transformed at that instant. Express this law by an equation.

19. We shall be able to proceed more quickly afterwards, and we shall be more certain that our work is correct, if we take a little time to think exactly what it is that we mean when we say that a function of h tends to a limit as h tends to zero. When in § 11 it is said that, as h (supposed positive) is diminished towards zero, $2x_A + h$ tends to $2x_A$ as a limit, it is meant that, without actually making $h = 0$, we can make $2x_A + h$ as near to $2x_A$ as we please. Thus if we want to make $2x_A + h$ differ from $2x_A$ by less than $\frac{1}{1000000}$ all we have to do is to make h less

than $\frac{1}{1000000}$. There is never any vagueness about a limit. The limit in this case is $2x_A$, not $2x_A +$ a very small fraction, or $2x_A -$ a very small fraction. If anyone thought it was $2x_A +$ a very small fraction f , we should only have to take h less than f to prove that $2x_A + h$ can be brought nearer to $2x_A$ than $2x_A + f$. Without taking h equal to 0 we can bring $2x_A + h$ as near to $2x_A$ as we please, but we cannot make $2x_A + h$ equal to $2x_A$.

In a case like this the limit is precisely known, but it is not a value of the function for any value which the variable can have. In fact $2x_A + h$ is a function of h , and h may have any value except 0, it cannot have the value 0 because we have divided by it. So $2x_A + h$ cannot have the value $2x_A$, but, as said before, it can be made to differ from this value by less than any number however small.

On the other hand there are cases in which the limit is a value of the function.

In the example which we considered the function was $2x_A + h$ and the limit was $2x_A$. The difference

$$(\text{function}) - (\text{limit})$$

may be positive or it may be negative. The property which distinguishes the limit from all other numbers is this:—The absolute value of the above difference (sign disregarded) can be made as small as may be wished by bringing the variable h near enough to zero.

20. As another example we consider the differentiation of x^3 . We have

$$(x+h)^3 = x^3 + 3hx^2 + 3h^2x + h^3,$$

so that

$$(x+h)^3 - x^3 = 3hx^2 + 3h^2x + h^3,$$

and

$$\frac{(x+h)^3 - x^3}{h} = 3x^2 + 3hx + h^2,$$

and we can show that the limit is $3x^2$. As we shall use a different method presently, it will be sufficient here to take the case where x and h are both positive. We have to show that, by taking h small enough, we can make $3hx + h^2$ as small as we please. We may begin by taking h to be smaller than x . Then $h^2 < hx$ and $3hx + h^2 < 4hx$. Let ϵ be any positive number as small as may be wished. We have to show that by taking h small enough we can

make $3hx + h^2$ less than ϵ . It appears that we need only take h to be less than $\frac{\epsilon}{4x}$ if this also makes h less than x . It will make h less than x , if ϵ is less than $4x^2$. As we want to show that we can make $3hx + h^2$ less than any number ϵ , however small, we may suppose ϵ to be less than $4x^2$. We take then ϵ to be less than $4x^2$, and h to be less than $\frac{\epsilon}{4x}$, and then $3hx + h^2$ is less than ϵ .

RULES OF DIFFERENTIATION

21. We need not go through the work of determining the appropriate limit in each case in order to differentiate, because we can reduce the process to the application of a few simple Rules.

(i) If $y = az$, where a is a number independent of x , and z is a function of x , we have the Rule

$$\frac{dy}{dx} = a \frac{dz}{dx}.$$

To prove the Rule we observe that, when x is changed to $x + \Delta x$, z is changed to $z + \Delta z$, and y to $y + \Delta y$, where $\Delta y = a\Delta z$. Hence $\frac{\Delta y}{\Delta x} = a \frac{\Delta z}{\Delta x}$. We are supposed to know that, as Δx tends to zero, $\frac{\Delta z}{\Delta x}$ tends to a limit, which is $\frac{dz}{dx}$. Therefore $\frac{\Delta y}{\Delta x}$ tends to a limit which is $a \frac{dz}{dx}$. But, if $\frac{\Delta y}{\Delta x}$ tends to a limit, that limit is $\frac{dy}{dx}$.

For example, we found that, when $s = t^2$, $\frac{ds}{dt} = 2t$, and, when $s = (16 \cdot 1) t^2$, $\frac{ds}{dt} = (32 \cdot 2) t$.

(ii) If $y = u + v$, where u and v are functions of x , we have the Rule

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

To prove the Rule we observe that, when x is changed to $x + \Delta x$, u is changed to $u + \Delta u$, v to $v + \Delta v$, y to $y + \Delta y$, where $\Delta y = \Delta u + \Delta v$. Hence $\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$. We are supposed to know that, as Δx tends to zero, $\frac{\Delta u}{\Delta x}$ tends to a limit, which is $\frac{du}{dx}$, and $\frac{\Delta v}{\Delta x}$ tends to a limit, which is $\frac{dv}{dx}$. Hence $\frac{\Delta y}{\Delta x}$ tends to a limit which is $\frac{du}{dx} + \frac{dv}{dx}$.

As an example, let $y = x^2 + x$. Then $\frac{dy}{dx} = 2x + 1$.

The Rule can be extended to a sum of n terms, n being any whole number, in the form :—The differential coefficient of a sum of terms is the sum of the differential coefficients of the terms.

(iii) If $y = uv$, where u and v are functions of x , we have the Rule

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}.$$

To prove the Rule we observe that

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ &= uv + u \Delta v + v \Delta u + \Delta u \Delta v, \end{aligned}$$

so that $\Delta y = u \Delta v + v \Delta u + \Delta u \Delta v$.

Hence $\frac{\Delta y}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \Delta v$.

We are supposed to know that, as Δx tends to zero, $\frac{\Delta v}{\Delta x}$ tends to a limit, which is $\frac{dv}{dx}$, and $\frac{\Delta u}{\Delta x}$ tends to a limit, which is $\frac{du}{dx}$, while Δv tends to zero. Hence the three terms on the right-hand side tend to limits which are $u \frac{dv}{dx}$, $v \frac{du}{dx}$, 0, and therefore the limit to which $\frac{\Delta y}{\Delta x}$ tends is $u \frac{dv}{dx} + v \frac{du}{dx}$.

This Rule is known as the Rule for “differentiating a product.”

Additional Rules of differentiation will be given in §§ 25 and 28 below. Some further discussion of the proofs of the Rules will be found in Appendix II.

22. We apply the Rule for differentiating a product to obtain the differential coefficient of x^n , where n is a positive integer.

We know that $\frac{d(x^2)}{dx} = 2x$, and $\frac{dx}{dx} = 1$.

Now $x^3 = x \cdot x^2$.

Put $x = u$, $x^2 = v$,

then $x^3 = uv$,

and $\frac{d(x^3)}{dx} = x \frac{d(x^2)}{dx} + x^2 \frac{dx}{dx} = x \cdot 2x + x^2 = 3x^2$.

Again $x^4 = x \cdot x^3$.

Put $x = u$, $x^3 = v$,

then $x^4 = uv$,

and $\frac{d(x^4)}{dx} = x \frac{d(x^3)}{dx} + x^3 \frac{dx}{dx} = x \cdot 3x^2 + x^3 = 4x^3$.

These results suggest the general formula

$$\frac{d(x^n)}{dx} = nx^{n-1}.$$

We have proved that this formula holds when $n = 2, 3, 4$, and we could go on in the same way to prove it for $n = 5, 6, \dots$. Let us suppose that it has been proved for all whole numbers $2, 3, 4, \dots$ up to some whole number k . We have

$$\frac{d(x^k)}{dx} = kx^{k-1}.$$

Put $x = u$, $x^k = v$,

then $x^{k+1} = uv$,

and

$$\begin{aligned}\frac{d(x^{k+1})}{dx} &= x \frac{d(x^k)}{dx} + x^k \frac{dx}{dx} \\ &= x k x^{k-1} + x^k \\ &= (k+1)x^k.\end{aligned}$$

If the formula holds for k it holds for $k+1$. But we proved it for 2, 3, 4. Hence it holds for all values of n which are positive integers and greater than 1.

The case where $n=1$ can be included by observing that $x^0=1$ and therefore the result $\frac{dx}{dx}=1$ can be written $\frac{dx}{dx}=1x^0$.

The result may also be written in the form $\frac{dy}{dx}=n\frac{y}{x}$ when $y=x^n$.

23. The formula for differentiating x^n (n a positive integer), combined with the rules (i) and (ii) of § 21, enables us to differentiate a large class of functions. A function of this class is expressed by a sum of terms, each term is the product of some constant number and a positive integral power of x . The general form of such a function is

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + px + q,$$

where $a, b, c, \dots p, q$ are numbers independent of x , and n is a positive integer. Such a function is called a "rational integral function of the n th degree." The function $mx + b$ which we considered in the last Chapter is a rational integral function of the first degree; it is often called a "linear" function, because its graph is a straight line.

If any term of a rational integral function has a minus sign prefixed to it, the differential coefficient has a minus sign prefixed to it. For example, if in the above expression ax^n were $-x^n$ the differential coefficient of that term would be $-nx^{n-1}$. The coefficient a is -1 .

EXAMPLES

1. Differentiate the following (1)–(14) :—

- (1) $-x^2$, (2) $\frac{1}{2}x^2$, (3) x^2+2x , (4) x^2-2x , (5) $\frac{1}{2}x^2+3x$,
 (6) $-\frac{1}{2}x^2+3x$, (7) $x(1-x)$, (8) $x^2(1+2x)$, (9) x^3-3x , (10) $\frac{1}{3}x^3$,
 (11) $-x^3$, (12) $2x^3-3x^2$, (13) x^3-4x^2+10x , (14) $\frac{1}{3}x^3-\frac{3}{5}x+\frac{1}{4}$.

2. The tangent at a point P to a curve, given by $ky=x^2$, where k is independent of x , meets the axis of x in T, and the ordinate of P meets the axis of x in N. Prove that OT is half of ON.

3. The curve is given by $ky=x^3$ and the notation is the same as in Ex. 2. Prove that OT is two-thirds of ON.

24. We shall show presently that the formula $\frac{d(x^n)}{dx} = nx^{n-1}$

holds in the cases where n is fractional and where n is negative. We call to mind the meanings of fractional and negative indices. We know that the meaning of an index in general is fixed by the equation

$$x^m x^n = x^{m+n},$$

called the "index law." We know that, if n is a positive integer, this law shows that x^n means the product of n factors, each equal to x . We know also that if n is a positive fraction

of the form $\frac{p}{q}$, where p and q are positive integers, this law shows

that x^n , or $x^{\frac{p}{q}}$, is the p th power of the q th root of x , and that it is also the q th root of x^p . If p and q have any common

divisor other than 1 it may be removed, so that the fraction $\frac{p}{q}$ may

be taken to be in its lowest terms, and when this is done, it is

understood that, if q is even, x is positive. If p is even $x^{\frac{p}{q}}$

is always positive. If p and q are both odd $x^{\frac{p}{q}}$ has the same

sign as x . Further the index law shows that, if n is negative, and we put $n = -m$, so that m is positive, x^n or x^{-m} means $\frac{1}{x^m}$.

25. We next introduce two additional Rules of differentiation. We number them consecutively with the Rules in § 21.

(iv) If we are given y as a function of z , and z as a function of x , we may regard y as a function of x . Then we have the Rule

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx}.$$

It is important to notice that we may not infer this by cancelling the dz 's, as if the expressions were fractions. In order to prove the Rule we must remember what the various expressions mean.

When x is changed to $x + \Delta x$, z is changed to $z + \Delta z$, and then y is changed to $y + \Delta y$. We are supposed to know that $\frac{\Delta y}{\Delta z}$ and

$\frac{\Delta z}{\Delta x}$ tend to limits when Δx tends to zero. These limits are $\frac{dy}{dz}$ and $\frac{dz}{dx}$. Hence the product $\frac{\Delta y}{\Delta z} \frac{\Delta z}{\Delta x}$ tends to a limit which is $\frac{dy}{dz} \frac{dz}{dx}$.

But in the product $\frac{\Delta y}{\Delta z} \frac{\Delta z}{\Delta x}$ we may cancel the Δz 's because the two factors are quotients, and therefore this product is equal to $\frac{\Delta y}{\Delta x}$. We know therefore that $\frac{\Delta y}{\Delta x}$ tends to a limit, and that this limit is $\frac{dy}{dz} \frac{dz}{dx}$.

This Rule is known as the Rule for differentiating a "function of a function."

(v) If y is a function of x , x is also a function of y . Sometimes it is easy to find $\frac{dx}{dy}$. Then we can write down the value of $\frac{dy}{dx}$ by the Rule $\frac{dy}{dx} \frac{dx}{dy} = 1$.

It is important to notice that the Rule may not be inferred by cancelling the dx 's and dy 's.

When x is changed to $x + \Delta x$, y is changed to $y + \Delta y$. Hence not only is Δy the number that is added to y when Δx is added to x , but also Δx is the number that is added to x when Δy is added to y . Therefore in the product $\frac{\Delta y}{\Delta x} \frac{\Delta x}{\Delta y}$ we may cancel the Δx 's and the Δy 's, and the value of the product is 1. Now we are supposed to know that as Δy tends to zero $\frac{\Delta x}{\Delta y}$ tends to a limit which is $\frac{dx}{dy}$. Hence $\frac{\Delta y}{\Delta x}$ tends to a limit which is $\frac{1}{\frac{dx}{dy}}$.

26. The variable with respect to which we differentiate is usually called the "independent variable," and a variable which is regarded as a function of the independent variable is usually called a "dependent variable." The Rule (iv) is the rule for changing the independent variable. The Rule (v) is the rule for interchanging the dependent and independent variables.

27. We can now verify the general formula

$$\frac{d(x^n)}{dx} = nx^{n-1}.$$

First let $n = \frac{p}{q}$, where p and q are positive integers.

Put

$$y = x^n = x^{\frac{p}{q}},$$

then

$$x^p = y^q,$$

and we may put

$$z = x^p, \quad z = y^q,$$

and then

$$\frac{dz}{dx} = px^{p-1}, \quad \frac{dz}{dy} = qy^{q-1}.$$

Now

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dz} \frac{dz}{dx} = \frac{1}{\frac{dz}{dy}} \cdot \frac{dz}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q} \cdot \frac{x^p}{y^q} \cdot \frac{y}{x} \\ &= \frac{p}{q} \frac{y}{x} = \frac{p}{q} x^{\frac{p}{q}-1} = nx^{n-1}. \end{aligned}$$

The formula is now verified for all positive integers and positive fractions.

Next let $n = -m$, where m is a positive number, integral or fractional.

Put
$$y = x^n = x^{-m} = \frac{1}{x^m}.$$

Then
$$x^m y = 1, \text{ and } \frac{d(x^m)}{dx} = mx^{m-1}.$$

Now we use the rule for differentiating a product, and find

$$\frac{d(x^m y)}{dx} = mx^{m-1} y + x^m \frac{dy}{dx}.$$

But, since $x^m y = 1$, $\frac{d(x^m y)}{dx} = 0$.

Hence
$$x^m \frac{dy}{dx} + mx^{m-1} y = 0,$$

or
$$\frac{dy}{dx} = -m \frac{y}{x} = -mx^{-m-1} = nx^{n-1}.$$

The formula is now verified for all integral and fractional values of n , positive or negative.

In applying the formula and the Rule (i) of § 21 to differentiate ax^n we must pay attention to the signs. For example, if we had to differentiate $-x^{-\frac{1}{2}}$, the result would be $\frac{1}{2}x^{-\frac{3}{2}}$.

The special case in which $n = -1$ is very important. It gives the result

$$\frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2}.$$

28. We consider some additional formulae of a general character, which, however, are hardly distinct enough from the previous Rules to be regarded as new Rules.

(i) Let y be given in the form $\frac{1}{u}$, where u is a function of x which we know how to differentiate. We have

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \text{and} \quad \frac{dy}{du} = -\frac{1}{u^2},$$

hence
$$\frac{dy}{dx} = -\frac{1}{u^2} \frac{du}{dx}.$$

(ii) If y is given in the form $\frac{v}{u}$, where v and u are functions of x , we may regard y as the product of v and $\frac{1}{u}$, and apply the rule for differentiating a product. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{u} \frac{dv}{dx} + v \frac{d}{dx} \frac{1}{u} = \frac{1}{u} \frac{dv}{dx} - v \frac{1}{u^2} \frac{du}{dx} \\ &= \frac{u \frac{dv}{dx} - v \frac{du}{dx}}{u^2}.\end{aligned}$$

This result is often called the rule for “differentiating a quotient.”

29. Among the functions which can be expressed in the form $\frac{v}{u}$ are those in which both v and u are rational integral functions of x . Such functions are described as “rational fractional functions” of x . Rational fractional functions and rational integral functions together constitute the class of “rational functions.” We know now how to differentiate every rational function. We know also how to differentiate many other functions, involving fractional powers. Such functions are not rational functions. It is necessary to practise differentiation without thinking about its applications, so as to be able to differentiate without making mistakes.

30. We work out some examples.

$$(i) \quad \frac{1}{\sqrt{1-x^2}}.$$

$$\text{Let } u = 1 - x^2, \text{ and let } y = u^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}.$$

$$\begin{aligned}\text{Then } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \left(-\frac{1}{2} u^{-\frac{3}{2}} \right) (-2x) = xu^{-\frac{3}{2}} \\ &= \frac{x}{(1-x^2)^{\frac{3}{2}}}.\end{aligned}$$

$$(ii) \quad x + \sqrt{(x^2 - 1)}.$$

Let $u = x^2 - 1$, and let $y = x + u^{\frac{1}{2}}$.

$$\begin{aligned} \text{Then} \quad \frac{dy}{dx} &= 1 + \frac{d(u^{\frac{1}{2}})}{dx} = 1 + \frac{d(u^{\frac{1}{2}})}{du} \frac{du}{dx} \\ &= 1 + \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(2x) = 1 + xu^{-\frac{1}{2}} \\ &= 1 + \frac{x}{\sqrt{(x^2 - 1)}}. \end{aligned}$$

This result may also be written

$$\frac{dy}{dx} = \frac{x + \sqrt{(x^2 - 1)}}{\sqrt{(x^2 - 1)}} = \frac{y}{\sqrt{(x^2 - 1)}}.$$

$$(iii) \quad x(1 - 2x)^2.$$

Let $y = x(1 - 2x)^2$. Apply the rule for differentiating a product. We get

$$\frac{dy}{dx} = (1 - 2x)^2 + x \frac{d\{(1 - 2x)^2\}}{dx}.$$

$$\begin{aligned} \text{Put } 1 - 2x = u, \text{ then } \frac{d\{(1 - 2x)^2\}}{dx} &= \frac{d(u^2)}{dx} = \frac{d(u^2)}{du} \frac{du}{dx} \\ &= 2u(-2) = -4(1 - 2x). \end{aligned}$$

$$\text{Hence} \quad \frac{dy}{dx} = (1 - 2x)^2 - 4x(1 - 2x) = (1 - 2x)(1 - 6x).$$

$$(iv) \quad \frac{x}{x^2 + 1}.$$

$$\text{Let} \quad v = x, \quad u = x^2 + 1, \quad y = \frac{v}{u}.$$

$$\text{Then} \quad \frac{dv}{dx} = 1, \quad \frac{du}{dx} = 2x,$$

$$\text{and therefore} \quad \frac{dy}{dx} = \frac{(x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(1 + x^2)^2}.$$

EXAMPLES

Differentiate the following (1)–(16), in which a, b, p, q denote numbers independent of x :—

- $$\begin{aligned} (1) \quad & \frac{1}{\sqrt{(1 - 2x^2)}}, \quad (2) \quad \frac{1}{\sqrt{(2 - x^2)}}, \quad (3) \quad \frac{x}{\sqrt{(3 - x^2)}}, \quad (4) \quad x + \sqrt{(x^2 + 2)}, \\ (5) \quad & \sqrt{(x^2 + 2)} - x, \quad (6) \quad x(13 - 2x)(15 - 2x), \quad (7) \quad x(a - 2x)(b - 2x), \\ (8) \quad & \frac{1}{1 + x}, \quad (9) \quad \frac{1}{1 - x}, \quad (10) \quad \frac{1}{x - 1}, \quad (11) \quad \frac{1 - x}{1 + x}, \quad (12) \quad \frac{x + 1}{\sqrt{(x^2 + 2x + 2)}}, \\ (13) \quad & \frac{x + 1}{\sqrt{(x^2 + x - 2)}}, \quad (14) \quad \frac{x + p}{(x + a)(x + b)}, \quad (15) \quad \frac{x + p}{\sqrt{(x^2 + 2px + q)}}, \\ (16) \quad & \frac{x - p}{\sqrt{(q + 2px - x^2)}}. \end{aligned}$$

THE SECOND DIFFERENTIAL COEFFICIENT

31. The differential coefficient $\frac{dy}{dx}$, or $f'(x)$, of a function y , or $f(x)$, with respect to x is itself a function of x , and we may differentiate this function. The differential coefficient of it is denoted by $\frac{d^2y}{dx^2}$, or $f''(x)$, and is called the "second differential coefficient" of y , or $f(x)$, with respect to x . The differential coefficient $\frac{dy}{dx}$ or $f'(x)$ is often called the "first differential coefficient."

It is important to regard the symbol $\frac{d^2y}{dx^2}$ as meaning $\frac{d\left(\frac{dy}{dx}\right)}{dx}$, and to remember what this means. We defined $\frac{dy}{dx}$ as the limit to which the quotient $\frac{\Delta y}{\Delta x}$ tends as Δx tends to zero. This limit $\frac{dy}{dx}$ has some value or other, say z , when x has a given value, and this z is a function of x . When x is changed to $x + \Delta x$, z is changed to $z + \Delta z$, and we may form the quotient $\frac{\Delta z}{\Delta x}$. This again tends to a limit when Δx tends to zero, and this limit is $\frac{dz}{dx}$,

$$\text{or } \frac{d\left(\frac{dy}{dx}\right)}{dx}, \text{ or } \frac{d^2y}{dx^2}.$$

When we have to calculate $\frac{d^2y}{dx^2}$ we must calculate $\frac{dy}{dx}$ first by differentiating y according to the Rules, and then we must use the Rules to find the differential coefficient of the new function $\frac{dy}{dx}$.

For example, in the case of the falling stone (§ 9), we have

$$s = (16 \cdot 1) t^2 \text{ and } \frac{ds}{dt} = (32 \cdot 2) t.$$

Now we differentiate this and we get

$$\frac{d^2s}{dt^2} = 32 \cdot 2.$$

This result is usually expressed by saying that the stone falls with a "constant acceleration," equal to 32·2 foot-second units of acceleration.

32. In general, if a body moving in a straight line passes over a distance s feet in t seconds, its velocity in feet per second is $\frac{ds}{dt}$. If we write v for $\frac{ds}{dt}$, v is a function of t , and $\frac{dv}{dt}$, if positive, measures the rate per second at which the velocity, measured in feet per second, is increasing. If $\frac{dv}{dt}$ is negative $-\frac{dv}{dt}$ measures the rate per second at which the velocity, measured in feet per second, is diminishing. In either case $\frac{dv}{dt}$, or $\frac{d^2s}{dt^2}$, is the measure in foot-second units of a certain quantity, which is called the "acceleration" in the direction in which s increases, whether the velocity is really increasing or diminishing. The magnitude of the second differential coefficient $\frac{d^2s}{dt^2}$ tells us the rate at which the velocity is changing, and the sign tells us whether it is increasing or diminishing.

If we construct a velocity-time graph by taking t and v as coordinates of a point, v being written for $\frac{ds}{dt}$, the gradient of the graph is equal to the measure of the acceleration.

33. The sign of the second differential coefficient $\frac{d^2y}{dx^2}$ admits of a simple graphic interpretation. If, as x increases, $\frac{dy}{dx}$ also

increases, $\frac{d^2y}{dx^2}$ is positive. If, as x increases, $\frac{dy}{dx}$ diminishes, $\frac{d^2y}{dx^2}$ is negative. In the first case the gradient of the graph of y as a function of x increases, in the second case it diminishes. If we take two points on a straight line parallel to the axis of y , one on the graph, and the other above it, the part of the curve near the lower point is concave to the upper point when $\frac{d^2y}{dx^2}$ is positive, convex to the upper point when $\frac{d^2y}{dx^2}$ is negative (Fig. 13).

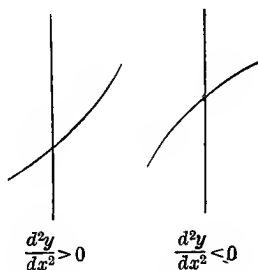


Fig. 13.

EXAMPLES

1. Find the second differential coefficients of the functions in the Examples on pp. 27 (Ex. 1) and 32.

2. A body moves in such a way that its acceleration at any instant is proportional to the distance that it has travelled at that instant. Express this fact by an equation.

3. A body moves in such a way that its acceleration at any instant is proportional to the distance that it still has to travel in order to reach a certain point, a feet from the starting point. Express this fact by an equation.

4. A body moves in such a way that its acceleration at any instant is proportional to its velocity at the instant. Express this fact by an equation.

5. Prove that the graph of $y = \alpha x^2 + \beta x + \gamma$, where α, β, γ are numbers independent of x , is concave or convex to points above it according as $\alpha >$ or < 0 .

6. Prove that the graph of $y = x^3$ is convex to points that are above it when x is negative, concave when x is positive.

CHAPTER III

SOME APPLICATIONS OF DIFFERENTIATION

34. BEFORE proceeding with the theory we consider some applications of the results so far obtained.

We have already seen what is meant by a function, and how a function may be represented by a graph. Instead of thinking of a function, such as x^2 , or an equation, such as $y = x^2$, and the corresponding graph, we may think about a curve, such as a circle or a parabola. The curve is the locus of a point which moves according to some law. For instance, a circle is the locus

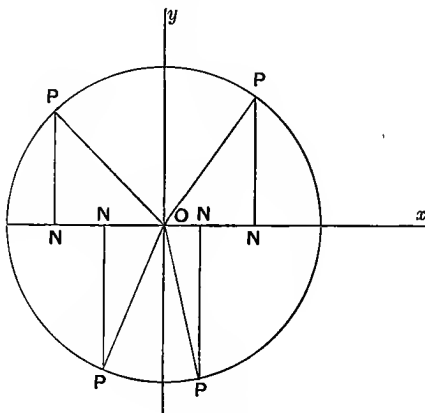


Fig. 14.

of a point which moves in such a way that its distance from a fixed point is always the same. To express this law, we take the fixed point as origin, and the distance of a point on the circle from the origin as r units of length, take any two straight lines drawn through the fixed point at right angles to each other as axes of x and y , and draw the ordinate PN of any point on the circle. Since the angle at N is always a right angle (Fig. 14), we have, in every position of P or (x, y) ,

$$x^2 + y^2 = r^2.$$

The parabola arises in connexion with the problem:—One side of a rectangle being given, it is required to find the other side so that the rectangle may be equal in area to a square. In Fig. 15 let AB be the given side, $ACDE$ the square, $ABMN$ the required rectangle, and let MN, CD meet in P . If we keep A and B fixed, and let C move on the line AB , the locus of P is a parabola. If we take the axes of x and y along AB and AE , and the length of AB as k units of length, we have the equation $ky = x^2$. If $k = 1$ we have $y = x^2$ as in § 9.

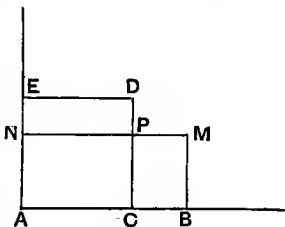


Fig. 15.

When, as in these examples, we express the geometrical condition, by which the points of a curve are distinguished from other points, by an equation connecting the coordinates of a point, this equation is called the “equation to the curve.”

35. When the equation to a parabola is given in the form

$$ky = x^2,$$

the axis of y is the “axis” of the parabola, the origin is the “vertex” of the parabola, and the axis of x is the “tangent at the vertex.” The curve lies above or below the tangent at the vertex according as k is positive or negative. The equation

$$k(y - y_0) = (x - x_0)^2$$

is the equation to a parabola, of which the axis is the straight line whose equation is $x=x_0$, and the vertex is the point (x_0, y_0) . Writing this equation

$$y = \frac{1}{k}x^2 - \frac{2x_0}{k}x + \frac{x_0^2}{k} + y_0,$$

we see that it has the form

$$y = ax^2 + \beta x + \gamma \dots\dots\dots(1),$$

where a, β, γ are independent of x . Conversely this equation can be written

$$\frac{1}{a} \left\{ y - \left(\gamma - \frac{1}{4} \frac{\beta^2}{a} \right) \right\} = \left(x + \frac{1}{2} \frac{\beta}{a} \right)^2,$$

and therefore every equation of the form (1) represents a parabola with its axis parallel to the axis of y .

In general we can find a parabola having its axis parallel to the axis of y , and passing through three given points. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the points. We have to find three numbers a, β, γ so that

$$ax_1^2 + \beta x_1 + \gamma = y_1, \quad ax_2^2 + \beta x_2 + \gamma = y_2, \quad ax_3^2 + \beta x_3 + \gamma = y_3 \dots\dots(2).$$

From these equations we find

$$y_2 - y_1 = a(x_2^2 - x_1^2) + \beta(x_2 - x_1), \quad y_3 - y_2 = a(x_3^2 - x_2^2) + \beta(x_3 - x_2),$$

and therefore

$$\frac{y_2 - y_1}{x_2 - x_1} = a(x_2 + x_1) + \beta, \quad \frac{y_3 - y_2}{x_3 - x_2} = a(x_3 + x_2) + \beta \dots\dots\dots(3),$$

whence

$$\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} = a(x_3 - x_1) \dots\dots\dots(4).$$

Equation (4) gives a . When a is found from it, either of the equations (3) gives β . When a and β are known either of the equations (2) gives γ .

It is assumed that no two of the given x 's are equal.

36. The equation to a curve expresses a geometrical property which is common to all the points of the curve. By means of the equation and some analytical process, e.g. differentiation, we may discover other properties of the curve. For example in the case of the parabola whose equation is $y = x^2$ we found that the tangent bisects the abscissa (§ 18).

37. The equation to a straight line expresses the condition that the straight line has the same gradient at every point.

The equation to the straight line which passes through the point $(0, b)$ and has the gradient m is

$$y = mx + b.$$

The equation to the straight line which passes through the point (x_1, y_1) and has the gradient m is

$$y - y_1 = m(x - x_1).$$

The equation to the straight line which passes through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

for the fraction $\frac{y_2 - y_1}{x_2 - x_1}$ is the gradient of the straight line (§ 7).

38. Two straight lines at right angles to each other being drawn through the origin, it is required to find an equation connecting their gradients.

The line Ox drawn to the right lies in one of the four angles formed by the two straight lines, and therefore one of them goes up to the right, and the other down to the right. Let the gradient of the one that goes up to the right be m , that of the other m' . Then m is a positive number, and m' is a negative number.

Let P be a point on the line that goes up to the right, PNQ a straight line at right angles to the axis of x , and let the x of P (and Q) be positive (Fig. 16). The angles OPN , NOP are together equal to a right angle, and so are the angles NOP , QON . Hence the angle OPN is equal to the angle QON . Also, in the triangles OPN , QON the angles at N are equal, being right angles. Therefore the triangles OPN , QON are similar, and

$$QN : ON = ON : PN.$$

Now the y of Q is negative, and $-y_Q$ is the number of units of length in the length of NQ . We have therefore

$$\frac{-y_Q}{x_Q} = \frac{x_P}{y_P}, \text{ or } -m' = \frac{1}{m}.$$

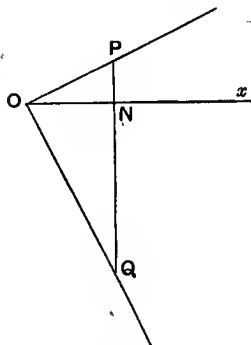


Fig. 16.

The required equation is

$$mm' = -1.$$

39. The equation connecting the gradients is the same for any two straight lines which are at right angles to each other, whether they pass through the origin or not. If

$$y - y_1 = m(x - x_1)$$

is the equation to a straight line passing through the point (x_1, y_1) ,

$$(y - y_1)m + x - x_1 = 0$$

is the equation to a straight line passing through the same point and cutting the other at right angles.

40. We can form the equation to the tangent to a curve at a point. Let $y = f(x)$ be the equation to the curve, x_1 and $f(x_1)$ the coordinates of the point. The gradient of the tangent is the limit to which the quotient

$$\frac{f(x_1 + h) - f(x_1)}{h}$$

tends as h tends to zero. This limit is obtained by substituting x_1 for x in the derived function $f'(x)$. We may write $f'(x_1)$ for this limit. Then the equation to the tangent is

$$y - y_1 = f'(x_1)(x - x_1).$$

To find $f'(x_1)$, first differentiate $f(x)$, then substitute x_1 for x .

A straight line drawn through a point of a curve at right angles to the tangent at the point is called the "normal" at the point.

If $y = f(x)$ is the equation to the curve, the gradient of the normal is $-\frac{1}{f'(x_1)}$, where $f'(x_1)$ is the gradient of the tangent, as has just been explained. The equation to the normal is therefore

$$(y - y_1)f'(x_1) + (x - x_1) = 0.$$

41. If $\frac{dy}{dx} = 0$ at a point of a curve we know that the tangent to the curve at the point is parallel to the axis of x . If the point is on the axis of x , the tangent is the axis of x . For

example, the tangent at the origin to the parabola whose equation is $y = x^2$ is the axis of x .

If $y = \sqrt{x}$ we have in general $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, but this formula does not determine the value of $\frac{dy}{dx}$ at the origin because it directs us to divide by 0, which we cannot do. This difficulty is met by observing that in general $\frac{dx}{dy} = 2\sqrt{x}$, and when $x = 0$ this equation becomes $\frac{dx}{dy} = 0$. The tangent at the origin to the curve whose equation is $y = \sqrt{x}$ is the axis of y .

Whenever the evaluation of $\frac{dy}{dx}$ would require division by 0, i.e. whenever $\frac{dx}{dy} = 0$ at a point of a curve, the tangent to the curve at the point is parallel to the axis of y . If the point is on the axis of y the tangent is the axis of y .

EXAMPLES

1. Prove that, if $y = ax^2 + \beta x + \gamma$ is the equation to a parabola passing through the points $(a - h, y_1)$, (a, y_2) , $(a + h, y_3)$, then

$$2ah^2 = y_1 + y_3 - 2y_2.$$

2. In the same case prove that

$$2\beta h^2 = 4y_2 a - y_1(2a + h) - y_3(2a - h),$$

$$2\gamma h^2 = y_1 a(a + h) + y_3 a(a - h) - 2y_2(a^2 - h^2).$$

3. Find the equations to the tangent and normal at any point (x_1, y_1) on the parabola whose equation is $ky = x^2$.

4. Find the equation to the tangent at any point (x_1, y_1) on the hyperbola whose equation is $y = \frac{1}{x}$. Prove that, if the tangent at P meets the axis of x in T and the axis of y in U, and if PN is the ordinate of P, N is the middle point of OT, and that P is the centre of a semicircle passing through T, O, U. (Fig. 17.)

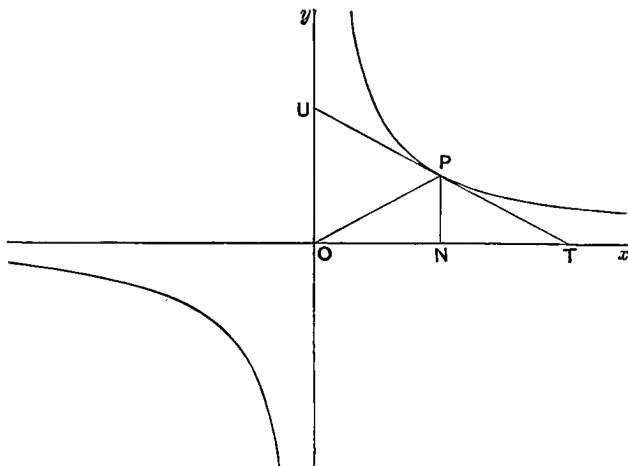


Fig. 17.

5. Prove that, if $y = \sqrt{r^2 - x^2}$ or $y = -\sqrt{r^2 - x^2}$, where r is independent of x , $\frac{dy}{dx} = -\frac{x}{y}$; and hence verify that the normal to a circle at any point passes through the centre of the circle.

APPROXIMATIONS

42. If we think what such a result as $\frac{d(x^n)}{dx} = nx^{n-1}$ means, we remember that it implies that

$$\frac{(x+h)^n - x^n}{h} \text{ is nearly equal to } nx^{n-1}$$

when h is very small. It is convenient to have a symbol for "is nearly equal to" and we shall use the symbol " \doteq ". Thus we write

$$(x+h)^n - x^n \doteq nx^{n-1}h,$$

or again

$$(x+h)^n \doteq x^n + nx^{n-1}h.$$

If in this equation we put $x = 1$ we get

$$(1 + h)^n \approx 1 + nh \dots \dots \dots (1)$$

This approximate equation is often useful. It gives an approximate value of the n th power of any number which differs very little from 1. For the approximation to be a good one it is necessary that n should not be too great.

43. The result (1) of § 42 can often be applied to the approximate extraction of square, cube, and other roots. As an example we find the fifth root of 31.

We have

$$\begin{aligned} (31)^{\frac{1}{5}} &= (32 - 1)^{\frac{1}{5}} = \left\{ 32 \left(1 - \frac{1}{32} \right) \right\}^{\frac{1}{5}} \\ &= (32)^{\frac{1}{5}} \left(1 - \frac{1}{32} \right)^{\frac{1}{5}} = 2 \left(1 - \frac{1}{32} \right)^{\frac{1}{5}} \\ &\approx 2 \left(1 - \frac{1}{160} \right). \end{aligned}$$

Hence

$$(31)^{\frac{1}{5}} \approx \frac{159}{80}, \text{ or } (31)^{\frac{1}{5}} \approx 1.9875.$$

The actual value correct to 5 places is 1.98734.

44. In another way of using the approximate equation (1) of § 42 we suppose that x times some unit is the measure of a variable quantity, and x^n times some unit is the measure of a related quantity. If the first quantity can be measured accurately to a units, where a is a small fraction, the true value of the first quantity may be anything between $x + a$ units and $x - a$ units. The true value of the related quantity may be anything between $(x + a)^n$ units and $(x - a)^n$ units, or $x^n \left(1 + \frac{a}{x} \right)^n$ and $x^n \left(1 - \frac{a}{x} \right)^n$ units. Hence the possible error of measurement of the related quantity is approximately nax^{n-1} times the appropriate unit.

45. Again the process of differentiating a product gives the approximate equation

$$\Delta(uv) \doteq v\Delta u + u\Delta v.$$

This result may be illustrated by a figure. The area of a rectangle of sides u inches and v inches is uv square inches. If small additions Δu inches and Δv inches are made to the sides the area is increased by $v\Delta u + u\Delta v$ square inches approximately.

In Fig. 18 the lengths of AB, AD can be taken to be u inches and v inches, and the lengths of BE, DG, Δu inches and Δv inches. The area of the rectangle DGKC is $u\Delta v$ square inches, and the area of the rectangle BEHC is $v\Delta u$ square inches. The actual increment

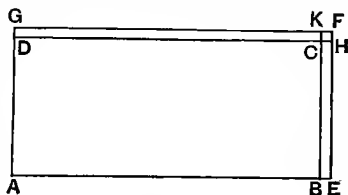


Fig. 18.

of area differs from that calculated by the approximate rule by the area of the rectangle KFHC, which is much smaller than either DGKC or BEHC.

46. More generally, since the limit to which

$$\frac{f(x+h) - f(x)}{h}$$

tends as h tends to zero is $f'(x)$, we have the approximate equation

$$f(x+h) \doteq f(x) + hf'(x).$$

47. As illustrating this approximate equation we consider the coefficient of linear expansion and the coefficient of cubical expansion of a substance, e.g. copper. If a rod of the substance is warmed it becomes longer. Let the initial length be l inches, and the initial temperature t degrees Centigrade.

When t is changed to $t + \Delta t$, l is changed to $l + \Delta l$, and, as Δt tends to zero, $\frac{\Delta l}{\Delta t}$ tends to a limit, which is $\frac{dl}{dt}$. Let $\frac{dl}{dt} = \alpha l$. Then α is called the "coefficient of linear expansion," and we have the approximate equation

$$l + \Delta l \doteq l(1 + \alpha \Delta t).$$

We see that, if $\Delta t = 1$, so that the temperature rises 1 degree Centigrade, α is nearly equal to the ratio of the increase of length to the original length.

If a lump of the substance, of volume v cubic inches, is warmed, so that its temperature rises from t degrees Centigrade to $t + \Delta t$ degrees Centigrade, v will be changed to $v + \Delta v$, and, as Δt tends to zero, $\frac{\Delta v}{\Delta t}$ tends to a limit, which is $\frac{dv}{dt}$. Let $\frac{dv}{dt} = \beta v$. Then β is called the "coefficient of cubical expansion," and we have the approximate equation

$$v + \Delta v \doteq v(1 + \beta \Delta t).$$

As before, if the temperature is raised 1 degree Centigrade, β is nearly equal to the ratio of the increase of volume to the original volume.

Now if the lump is a cube, of side l inches, $v = l^3$, and

$$\frac{dv}{dt} = 3l^2 \frac{dl}{dt},$$

or $\beta l^3 = 3al^3$, so that $\beta = 3\alpha$. We have the result that the coefficient of cubical expansion is 3 times the coefficient of linear expansion.

EXAMPLES

1. Find approximately by the method of § 43 the values of $(28)^{\frac{1}{3}}$ and $(80)^{\frac{1}{4}}$.

2. The side of a cube can be measured accurately to $\frac{1}{100}$ of an inch, and the side is measured as 10 inches. Find approximately the possible error of measurement of the volume.

3. The sides of a rectangle can be measured accurately to $\frac{1}{100}$ of an inch, and the perimeter is measured as 125 inches, find approximately the possible error of measurement of the area.

4. The coefficient of linear expansion of copper is 0.0000167. By how much is a copper rod, 1 foot in length at 0° Centigrade, extended when its temperature is raised 10° Centigrade? If the area of the section of the rod at the lower temperature is 1 square inch, what is its area at the higher temperature?

MAXIMA AND MINIMA

48. As an example of a different kind of application of the Differential Calculus we take the problem of finding the shortest possible perimeter of a rectangle of given area.

Let the area be a square inches, and let y inches be the length of the perimeter, and x inches the length of one side. The length of the opposite side also is x inches, and the lengths of the other two sides are each $\left(\frac{1}{2}y - x\right)$ inches. We have

$$x\left(\frac{1}{2}y - x\right) = a,$$

or
$$y = 2\left(x + \frac{a}{x}\right).$$

Now
$$\frac{dy}{dx} = 2\left(1 - \frac{a}{x^2}\right).$$

Hence, if $x^2 < a$, $\frac{dy}{dx}$ is negative, the graph of y as a function of x goes down to the right. If $x^2 > a$, $\frac{dy}{dx}$ is positive, the graph of y as a function of x goes up to the right. If we begin with any small value of x , and let x gradually increase, y at first diminishes, and it goes on diminishing until $x^2 = a$, or $x = \sqrt{a}$; then y begins to increase, and it goes on increasing however great we make x . It follows that y has its least value when $x = \sqrt{a}$. This least value is found, by substituting \sqrt{a} for x in the expression $2\left(x + \frac{a}{x}\right)$, to be $4\sqrt{a}$. We see that the rectangle of given area and shortest perimeter is the square which has the given area.

49. We consider another problem of this kind. Suppose that we have a piece of cardboard in the form of a square of side 1 foot, and that we propose to make a box without a lid by turning up the edge of the square all round. The problem is to make the box of greatest volume. The full lines in Fig. 19 show the edges of the square, and the dotted lines show lines along which the cardboard may be bent. We take the height of the box to be x feet, ($x < 1$). We see that we get a box with a square base of side

1 - 2x feet and a height x feet, and, if the volume of the box is y cubic feet, y is given by the equation

$$y = (1 - 2x)^2 x.$$

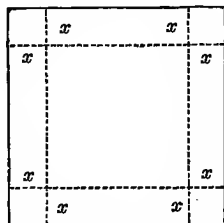


Fig. 19.

If we take x very small, we get a very shallow box and its volume is very small; if we take x nearly equal to $\frac{1}{2}$, we get a deep slender box, and the volume is again very small. Now (Ex. iii in § 30)

$$\frac{dy}{dx} = (1 - 2x)(1 - 6x) = 12 \left(\frac{1}{2} - x \right) \left(\frac{1}{6} - x \right).$$

When x is very small, both the factors on the right are positive, $\frac{dy}{dx}$ is positive, and as x increases up to $\frac{1}{6}$, y increases. When x lies between $\frac{1}{6}$ and $\frac{1}{2}$ the first factor is positive and the second negative, $\frac{dy}{dx}$ is negative, and as x increases from $\frac{1}{6}$ to $\frac{1}{2}$, y diminishes. The greatest value of y occurs when $x = \frac{1}{6}$, and this greatest value of y is $\frac{2}{27}$. The box of greatest volume is 2 inches deep and its volume is 128 cubic inches.

50. These problems lead us to some general considerations concerning the application of the Differential Calculus to questions of *maxima* and *minima*. If $y = f(x)$, then so long as $f'(x)$ is

positive, y increases as x increases; when $f'(x)$ is negative, y diminishes as x increases. At a point on a graph, where y changes from increasing to diminishing, or from diminishing to increasing, $f'(x)$ vanishes. At such a point the gradient of the graph is zero, and the tangent to the graph is parallel to the axis of x . When y changes from increasing to diminishing it has a

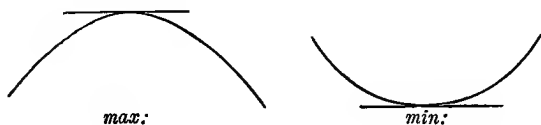


Fig. 20.

maximum value. When it changes from diminishing to increasing it has a *minimum* value. We can find the values of x which correspond to the maximum and minimum values of y if we can solve the equation $f'(x) = 0$, and we can find the maximum and minimum values of y by substituting in $f(x)$ the numbers that satisfy this equation.

51. It may happen, as in the problem of the box, that the equation $f'(x) = 0$ is satisfied by more than one value of x , and then we have to choose the right value. In simple problems, such as we are likely to meet with, we can always do this. For example in the problem of the box we see that the equation is satisfied by $x = \frac{1}{2}$ as well as by $x = \frac{1}{6}$. But we saw that $\frac{1}{6}$ was the right value, because when $x < \frac{1}{6}$, y is increasing, and when $x > \frac{1}{6}$, y is diminishing. We might have settled the point also by observing that when $x = \frac{1}{2}$, $y = 0$, and zero volume cannot be the maximum in such a problem.

52. The point can often be settled by using the second differential coefficient (§ 31).

We know that, when $f'(x)$ is positive, $f(x)$ increases as x increases, and when $f'(x)$ is negative, $f(x)$ diminishes as x increases. We know also that, if $f(x)$ has a maximum or a minimum value, $f'(x)$ vanishes for the corresponding value of x .

If $f''(x)$ is positive when $f'(x)$ vanishes, $f'(x)$ is increasing with x ; and, therefore, in passing through zero, $f'(x)$ passes from negative values to positive values. Let a be a value of x for which $f'(x)$ vanishes and $f''(x)$ is positive. When x is a little less than a , $f'(x)$ is negative, and $f(x)$ diminishes as x increases towards a . When x is a little greater than a , $f'(x)$ is positive, and $f(x)$ increases as x increases above a . Hence, as x increases through a , $f(x)$ changes from diminishing to increasing. Therefore a is a value of x for which $f(x)$ is a minimum.

In like manner, if b is a value of x for which $f'(x)$ vanishes and $f''(x)$ is negative, it is a value for which $f(x)$ is a maximum.

For example, in the problem of § 49

$$f(x) = x(1-2x)^2, \quad f'(x) = 1-8x+12x^2,$$

$$f''(x) = -8(1-3x).$$

In this case $f'(x)$ vanishes if $x = \frac{1}{6}$ or $\frac{1}{2}$. When $x = \frac{1}{6}$, $f''(x)$ is negative, and when $x = \frac{1}{2}$, $f''(x)$ is positive. Hence $x = \frac{1}{6}$ makes $f(x)$ a maximum, and $x = \frac{1}{2}$ makes $f(x)$ a minimum.

53. The length of the normal drawn from a point to a curve affords a good example of maxima and minima. Let P be a point on a curve, N a point on the normal at P . With centre N and radius NP describe a circle. Join N to Q a point of the curve near to P . If the part of the curve which is near to P is concave to N but outside the circle, NQ is greater than NP , and then NP is a straight line of minimum length drawn from the point N to the curve (Fig. 21 α). If the part of the curve which is near to P is concave to N but within the circle, NQ is less than NP , and NP is a straight line of maximum length drawn from N to the curve (Fig. 21 β). If the part of the curve which is near to P is convex to N , NP is a straight line of minimum length drawn from N to the curve (Fig. 21 γ).

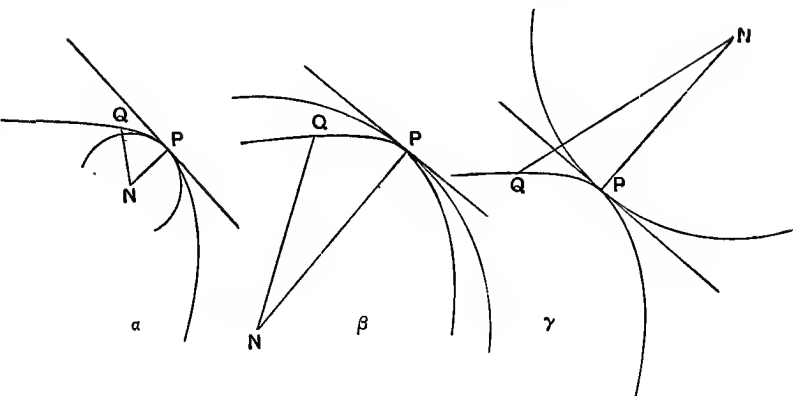


Fig. 21.

EXAMPLES

1. The perimeter of a rectangle is given. Find its shape so that its area may be as great as possible.
2. The length of the diagonal of a rectangle is given. Find its shape in order that (i) its perimeter, (ii) its area, may be as great as possible.
3. Solve the problem of the box (§ 49) when the piece of cardboard has the shape of a rectangle of sides 13 inches and 15 inches. [Result, depth = 2.3155 inches approximately.]
4. Solve the problem of the box when the sides of the rectangle are a inches and b inches.
5. According to the regulations of the Parcel Post the largest parcels that may be sent have such dimensions that the sum of the length and the girth does not exceed 6 feet. Find the dimensions of the largest parcel in the shape (i) of a prism on a square base, (ii) of a cylinder on a circular base. [Result, in both cases, length = 2 feet.] Find the volumes of both parcels.
6. Determine the shape of a right circular cone of given volume in order that its surface may be the least possible. [Result, height of cone : radius of base = 1.4142 approximately.]

THE THEOREM OF INTERMEDIATE VALUE

54. If a function $f(x)$ vanishes when x has the value a and also when x has the value b , the derived function $f'(x)$ must vanish for some value of x between a and b . This result is illustrated in Figs. 22 α and 22 β .

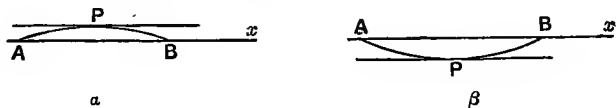


Fig. 22.

In Fig. 22 α the graph of $f(x)$ goes above the axis of x between A and B, and $f(x)$ has a maximum value at some intermediate point P. In Fig. 22 β the graph of $f(x)$ goes below the axis of x between A and B, and $f(x)$ has a minimum value at some intermediate point P. The function $f(x)$ may, of course, have more than one maximum or minimum between $x=a$ and $x=b$. It is certain that it has at least one.

55. This result is of very great importance in the more advanced portions of the Differential Calculus. We may give it a slightly more general form by remarking that, if A and B are two points on the graph of a function, there is on the graph between A and B a point P at which the tangent to the graph is parallel to AB (Fig. 23).

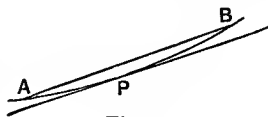


Fig. 23.

56. This second form of the result is reducible to the first form. If the graph is that of $f(x)$, and the gradient of the secant AB is m , we have

$$\frac{f(b) - f(a)}{b - a} = m,$$

where a and b are the x -coordinates of A and B. Now write

$F(x)$ for the function $f(b) - f(x) - (b-x)m$. Then $F(x)$ vanishes when $x = b$. Also, by the definition of m , $F(x)$ vanishes when $x = a$. Hence $F'(x)$ vanishes for some value of x between a and b . But $F'(x) = -f'(x) + m$.

Therefore $f'(x) = m$ for some value of x between a and b .

We have the result¹ that

$$\frac{f(b) - f(a)}{b - a} = f'(x)$$

for some value of x between a and b .

57. The result may also be written

$$\frac{f(x+h) - f(x)}{h} = f'(x'),$$

where x' is some number between x and $x+h$. The formula

$$\frac{f(x+h) - f(x)}{h} \approx f'(x)$$

noted in § 46 is an approximate equivalent of the above exact equation.

EXAMPLES

1. Find the value of x which satisfies the equation

$$f(b) - f(a) = (b-a)f'(x),$$

(i) when $f(x) = x^2$, (ii) when $f(x) = x^3$.

2. Interpret the equation

$$f(b) - f(a) = (b-a)f'(x)$$

as showing that the average velocity of a moving body in any interval is the same as the velocity at some instant during the interval.

If the body moves over s feet in t seconds, and $s = at^2$, where a is constant, the instant in question is the middle instant of the interval. If $s = \beta t^3$, where β is constant, the instant in question is always later than the middle instant of the interval. Prove these statements.

¹ For the necessary limitation of the result see Appendix II.

3. The theorem that, if $f(x)$ vanishes when $x=a$ and when $x=b$, $f'(x)$ vanishes for an intermediate value of x , may sometimes be applied to determine the number and situation of the real roots of an equation. For example the equation $2x^3+3x^2+6x-10=0$ has only one real root. Prove this.

4. Prove that the equation $x^3+px^2+qx+r=0$, in which p, q, r are independent of x , cannot have more than one real root if $p^2 < 3q$.

CHAPTER IV

INTEGRATION

58. WE consider some known results in regard to mensuration.

(a) If the length of one side of a triangle is b units of length, and the length of the perpendicular let fall upon this side from the opposite vertex is p units of length, the area of the triangle is $\frac{1}{2} bp$ units of area.

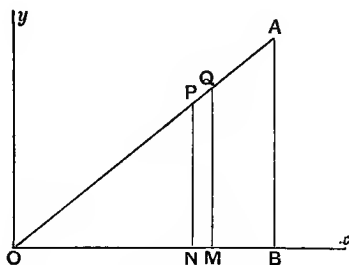


Fig. 24.

Consider a right-angled triangle OAB (Fig. 24). Let the lengths of OB, AB be p , b units of length. Take O as origin and the axis of x along OB. Let x , y be the coordinates of any

point P on OA, let PN be the ordinate of P, and let the area of the triangle OPN be z units of area. Then

$$z = \frac{1}{2} xy.$$

Since the triangles OAB, OPN are similar we have

$$\frac{y}{x} = \frac{b}{p},$$

and therefore

$$z = \frac{1}{2} \frac{b}{p} x^2.$$

This equation expresses z as a function of x . Since b and p are independent of x , we have

$$\frac{dz}{dx} = \frac{b}{p} x = y.$$

When P is moved along OA to Q, so that x is changed to $x + \Delta x$, y becomes $y + \Delta y$, and z becomes $z + \Delta z$. The area of the trapezium PNMQ is Δz units of area, and we have

$$\begin{aligned} \Delta z &= \frac{1}{2} \frac{b}{p} (x + \Delta x)^2 - \frac{1}{2} \frac{b}{p} x^2 \\ &= \frac{b}{p} \Delta x \left(x + \frac{1}{2} \Delta x \right). \end{aligned}$$

Now

$$\frac{y + \Delta y}{x + \Delta x} = \frac{b}{p} = \frac{y}{x},$$

and therefore

$$\Delta y = \frac{b}{p} \Delta x.$$

Hence

$$\begin{aligned} \Delta z &= \Delta x \left(y + \frac{1}{2} \Delta y \right) \\ &= \frac{1}{2} \{ y + (y + \Delta y) \} \Delta x. \end{aligned}$$

This is the known result that the area of the trapezium PNMQ is equal to that of a rectangle whose sides are MN and half the sum of PN and QM. We observe that Δz lies between $y\Delta x$ and $(y + \Delta y)\Delta x$.

(b) Let the radius of a circle be r units of length. Then the length of the circumference is $2\pi r$ units of length, where π is

a certain number which is approximately equal to 3.1416. The area of the circle is πr^2 units of area¹.

Let

$$z = \pi r^2,$$

then

$$\frac{dz}{dr} = 2\pi r.$$

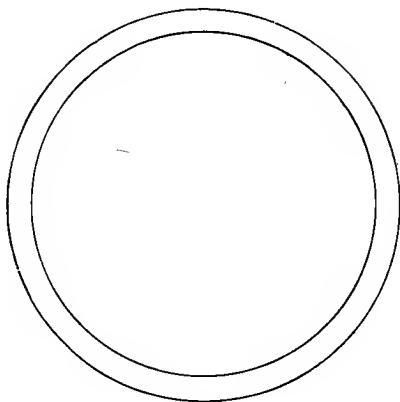


Fig. 25.

If the radii of two concentric circles (Fig. 25) are r and $r + \Delta r$ units of length, the area of the figure contained between them is $\pi \{(r + \Delta r)^2 - r^2\}$ units of area. When r is changed to $r + \Delta r$, z is changed to $z + \Delta z$, and

$$\Delta z = 2\pi \left(r + \frac{1}{2} \Delta r \right) \Delta r,$$

so that Δz lies between $2\pi r \Delta r$ and $2\pi (r + \Delta r) \Delta r$. The area of the included figure lies between the areas of two rectangles whose sides are, for one, the difference of the radii and the inner circumference, for the other, the difference of the radii and the outer circumference.

59. In the work that we have done so far we have had to differentiate given functions, but in many applications of the

¹ A discussion of the mensuration of the circle will be found in Appendix V.

Calculus we know the differential coefficient of a function before we know the function.

We consider some examples.

(a) *Area under a curve.* We consider a curve such as AB in Fig. 26, we take x, y to be the coordinates of a point P on the curve, and draw the ordinate PN. The area ACNP, bounded by AC, the axis of x , PN, and the arc AP, may be taken to be z

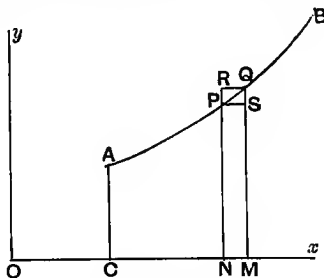


Fig. 26.

units of area. If we move P along the curve we change x , and also y , and we change z as well. We can regard z as a function of x . Usually we do not know what function z is, but we can find $\frac{dz}{dx}$.

When P is moved to Q, N is moved to M, and we may express the length of NM as Δx units of length. Also the length of QM is $y + \Delta y$ units of length. The area ACNP is increased by the area PNMQR, and this area is Δz units of area. Now, as the figure is drawn¹, this area is less than that of the rectangle QMNR, but greater than that of the rectangle PNMS, and the areas of these rectangles are respectively $(y + \Delta y) \Delta x$ and $y \Delta x$ units of area.

Hence Δz lies between $y \Delta x$ and $(y + \Delta y) \Delta x$, and $\frac{\Delta z}{\Delta x}$ lies

¹ In the figure the ordinate of Q is the greatest, and that of P is the smallest, in the arc PQ. The more general case is considered in Ch. V, § 75.

between y and $y + \Delta y$. Now when Δx tends to zero, Δy also tends to zero, and $\frac{\Delta z}{\Delta x}$ tends to a limit, which is y . But when $\frac{\Delta z}{\Delta x}$ tends to a limit, as Δx tends to zero, that limit is the differential coefficient $\frac{dz}{dx}$.

Hence we have the equation

$$\frac{dz}{dx} = y,$$

exactly as in § 58 (a).

If we know the equation to the curve we can express y in terms of x , and therefore we know $\frac{dz}{dx}$ in terms of x .

(b) *Volume of a part of a sphere.* We consider the volume of the portion of a sphere contained between two parallel planes, one plane passing through the centre of the sphere. We take the centre of the sphere as origin, and draw the axis of x at

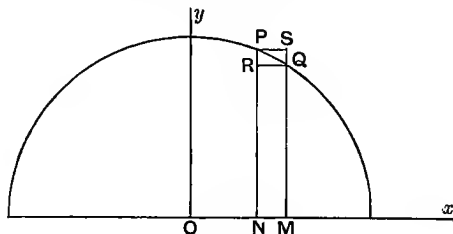


Fig. 27.

right angles to the planes. The surface of the sphere is generated by rotating a semicircle about its bounding diameter. Let P be any point on the semicircle (Fig. 27), we take it to the right of Oy ; and let PN be the ordinate of P . The section of the sphere by the plane which passes through Oy and is at right angles to Ox is a circle, and the section by the plane which passes through PN and is at right angles to Ox is another circle. Let x, y be the

coordinates of P, and let the volume of the portion contained between the two planes be v units of volume. Then v is a function of x . We do not know what function v is, but we can find $\frac{dv}{dx}$.

When P is moved to Q on the semicircle, x becomes $x + \Delta x$, y becomes $y + \Delta y$, and v becomes $v + \Delta v$. We note that when Δx is positive Δy is negative. The volume of the portion contained between the planes, which pass through PN and QM and are at right angles to OX, is Δv units of volume.

Complete the rectangles PNMS, QMNR. As the figure rotates each of them traces out a slice of a cylinder. The cylinder traced out by PNMS stands on a circular base whose area is πy^2 units of area, and its height is Δx units of length. The cylinder traced out by QMNR stands on a circular base whose area is $\pi (y + \Delta y)^2$ units of area, and its height also is Δx units of length. The volume Δv units of volume lies between the volumes of these two cylinders. Remembering that Δy is negative we have

$$\Delta v < \pi y^2 \Delta x, \text{ but } \Delta v > \pi (y + \Delta y)^2 \Delta x,$$

and therefore

$$\frac{\Delta v}{\Delta x} \text{ lies between } \pi y^2 \text{ and } \pi (y + \Delta y)^2.$$

Now, when Δx tends to zero, Δy also tends to zero, and $\frac{\Delta v}{\Delta x}$ tends to a limit, which is $\frac{dv}{dx}$.

Therefore
$$\frac{dv}{dx} = \pi y^2.$$

If the radius of the sphere is a units of length we have

$$x^2 + y^2 = a^2.$$

Hence
$$\frac{dv}{dx} = \pi (a^2 - x^2),$$

so that we know $\frac{dv}{dx}$ as a function of x .

60. These examples are sufficient to show the importance of the problem: the differential coefficient of a function being given, it is required to find the function.

In the first place let the differential coefficient of a function of x be zero for all values of x . Let y be the function.

We have
$$\frac{dy}{dx} = 0$$

for all values of x . If we had a graph of the function, the gradient of the graph would be zero everywhere. Now if y were increasing at any point, the gradient of the graph at that point would be positive; if y were diminishing at any point, the gradient of the graph at that point would be negative. Hence y never increases and it never diminishes. Now the values of y at two points cannot be different without y either increasing or diminishing between the points. Hence the value of y is the same at every point. In other words y is the same for all values of x , it is independent of x , or we have the result that y is a constant.

We can write the result $y = C$, where C means a "number independent of x ." The equation $\frac{dy}{dx} = 0$ is satisfied by putting $y = C$, whatever value (independent of x) we give to C , and it cannot be satisfied in any other way.

Since any constant value may be given to C , it is often called an *arbitrary* constant.

Next let the differential coefficient be 1. As before let y be the function. We have

$$\frac{dy}{dx} = 1$$

for all values of x . Now one way of satisfying this equation is to put $y = x$, but we cannot say that this is the only way. If possible let some other function y , different from x , satisfy the equation. We may write z for the difference $y - x$. Then we have

$$y = x + z, \text{ and } \frac{dy}{dx} = 1,$$

but
$$\frac{dy}{dx} = 1 + \frac{dz}{dx};$$

hence
$$\frac{dz}{dx} = 0.$$

It follows that z must be a constant. As before it is an arbitrary constant; denoting it by C , we have

$$y = x + C.$$

As a third example, we take the differential coefficient to be $2x$. We have

$$\frac{dy}{dx} = 2x$$

for all values of x . The equation is satisfied by putting

$$y = x^2,$$

and, as before, the general solution is of the form

$$y = x^2 + C,$$

where C is an arbitrary constant.

In the second and third examples we recognize the given value of $\frac{dy}{dx}$ as a function which we have found to be the differential coefficient of a particular function. Thus our problem is solved in two steps: first recognize the given differential coefficient as that of a particular function, and secondly add an arbitrary constant to this function.

61. When we recognize a function $f(x)$ as the differential coefficient of some particular function $\phi(x)$ we are said to "integrate" the function $f(x)$, and we call $\phi(x)$ an "integral" of $f(x)$. An integral of $f(x)$ is written

$$\int f(x) dx.$$

An integral of $f(x)$ means "a function of which $f(x)$ is the differential coefficient." The methods of finding integrals constitute the Integral Calculus.

62. If we can recognize a function $F(x)$ as one that has $f'(x)$ for its differential coefficient we write down the equation

$$\int f'(x) dx = F(x)$$

without adding a constant. But if we want to know what functions have $f'(x)$ as differential coefficient, in order that we may choose the right function in the solution of any problem, we write down the integral in the form $F(x) + C$. We shall see in the next Chapter how in special cases the constant C may be determined.

63. We take up the problem of integrating a given function without thinking about any applications.

Whenever we differentiate a function we integrate another function. For example

$$\frac{d(x^2)}{dx} = 2x \quad \text{and} \quad \int 2x dx = x^2$$

are different ways of writing the same result.

But it is hopeless to attempt to differentiate every conceivable function and tabulate the results so as to have a complete table of integrals. We have to proceed by recognizing the function to be integrated as being of a certain *form*, and tabulating the results for special forms of functions. In doing this we utilise the Rules of differentiation. See §§ 21 and 25 in Ch. II.

(i) The first Rule gives the result

$$\int a f'(x) dx = a \int f'(x) dx.$$

(ii) The second Rule gives the result

$$\int \{f'(x) + F(x)\} dx = \int f'(x) dx + \int F(x) dx,$$

or, more generally the integral of the sum of any number of functions is the sum of the integrals of the functions.

64. We consider next the result which we can obtain from the formula

$$\frac{d(x^n)}{dx} = nx^{n-1}.$$

This gives us $\int nx^{n-1} dx = x^n,$

or, by the rule (i), $\int x^{n-1} dx = \frac{x^n}{n}.$

In this formula change n into $n + 1$, we get

$$\int x^n dx = \frac{x^{n+1}}{n+1}. \quad (\text{A})$$

This formula holds for all values of n , positive or negative, integral or fractional, except $n = -1$. It is the first of a series of formulae which are known as *standard forms* because many other integrals can be evaluated by means of them.

By combining the result (A) with the rules already stated we see that we can integrate any function which can be expressed as a sum of terms, each term being of the form ax^n , where a is any constant, and n is any number positive or negative, integral or fractional, except -1 .

We shall find $\int x^{-1} dx$ in Ch. VI.

EXAMPLES

Integrate with respect to x the following (1)—(7):—

$$\begin{array}{llll} (1) \ x^2 + 3x - 1, & (2) \ 1 + 3x - x^2, & (3) \ \frac{1}{\sqrt{x}}, & (4) \ \frac{1}{x^2}, \\ (5) \ -\frac{1}{x^2} + 2\sqrt{x}, & (6) \ \frac{1}{4}x^{\frac{2}{3}}, & (7) \ \frac{2}{3}x^{\frac{1}{3}} - 2x^{-\frac{1}{2}}. \end{array}$$

The result in each case can be verified by differentiation.

65. The reduction of integrals to standard forms is generally to be effected by changing the variable and using the Rule for differentiating a function of a function (§ 25).

We take as an example

$$\int \frac{1}{(x+1)^2} dx.$$

We have to find a function y which satisfies the equation

$$\frac{dy}{dx} = \frac{1}{(x+1)^2}.$$

Now, if we put $z = (x+1)$,

we know a function of z which has $\frac{1}{z^2}$, that is $\frac{1}{(x+1)^2}$, as its differential coefficient with respect to z . The function is $-\frac{1}{z}$.

We try to find y as a function of z .

$$\text{We have } \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx}, \quad \frac{dy}{dz} = \frac{1}{(x+1)^2} = \frac{1}{z^2}, \quad \frac{dz}{dx} = 1,$$

$$\text{hence } \frac{dy}{dz} = \frac{1}{z^2},$$

$$\text{and } y = \int \frac{1}{z^2} dz = -\frac{1}{z} = -\frac{1}{x+1},$$

$$\text{or } \int \frac{1}{(x+1)^2} dx = -\frac{1}{x+1}.$$

66. As a more general example we take

$$\int (ax+b)^n dx,$$

where a and b are independent of x .

$$\text{Let } y = \int (ax+b)^n dx, \quad z = ax+b;$$

$$\text{we have } \frac{dy}{dz} = (ax+b)^n = z^n, \quad \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx}, \quad \frac{dz}{dx} = a.$$

Hence
$$\frac{dy}{dz} = \frac{1}{a} z^n,$$

and
$$y = \int \frac{1}{a} z^n dz = \frac{z^{n+1}}{a(n+1)},$$

or
$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)},$$

provided n is not -1 .

67. Another instructive example is

$$\int x \sqrt{x^2 + 1} dx.$$

Let
$$y = \int x \sqrt{x^2 + 1} dx, \quad z = x^2 + 1,$$

we have
$$\frac{dy}{dz} = x \sqrt{x^2 + 1} = xz^{\frac{1}{2}}, \quad \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx}, \quad \frac{dz}{dx} = 2x.$$

Hence
$$\frac{dy}{dz} = \frac{1}{2x} xz^{\frac{1}{2}} = \frac{1}{2} z^{\frac{1}{2}},$$

and
$$y = \frac{1}{2} \int z^{\frac{1}{2}} dz = \frac{1}{2} \frac{z^{\frac{3}{2}}}{\frac{3}{2}} = \frac{z^{\frac{3}{2}}}{3}.$$

or
$$\int x \sqrt{x^2 + 1} dx = \frac{1}{3} (x^2 + 1)^{\frac{3}{2}}.$$

68. In general let

$$y = \int f(x) dx,$$

and let z be a function of x . We have

$$\frac{dy}{dx} = f(x), \quad \frac{dy}{dz} = \frac{dy}{dx} \frac{dx}{dz}.$$

Hence
$$\frac{dy}{dz} = \frac{f(x)}{\frac{dz}{dx}} = f(x) \frac{dx}{dz}.$$

Now if we can express $f(x) \frac{dx}{dz}$ as a function of z , say $F(z)$, we have

$$y = \int F(z) dz.$$

EXAMPLES

Integrate with respect to x the following (1)—(8):—

$$(1) \frac{1}{(1-x)^2}, \quad (2) \frac{1}{(1-2x)^3}, \quad (3) \sqrt{2+3x}, \quad (4) \sqrt{2-3x},$$

$$(5) \frac{1}{\sqrt{2-3x}}, \quad (6) x\sqrt{1-x^2}, \quad (7) \frac{x}{\sqrt{1-x^2}},$$

$$(8) \frac{x}{\sqrt{1+x^2}}.$$

In Ex. (1)—(5) it is better to work out the result in each case by making the appropriate substitution, in the way explained in § 66, than to write it down by substituting in the formula there obtained. It is the method, not the formula, that is important. Any one who has once grasped the method has no need of the formula. The results in all cases can be verified by differentiation, and it is therefore unnecessary to record them here.

CHAPTER V

SOME APPLICATIONS OF INTEGRATION

69. WE consider an example of the determination of the area under a curve (§ 59 *a*)).

Let the curve be the graph of $y = x^2$, and let any straight line PQ be drawn across it parallel to the axis of x . Let (x, y) be

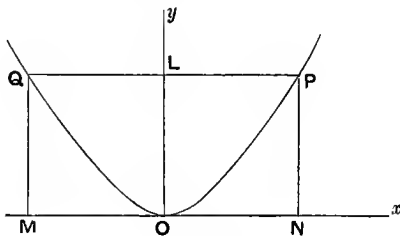


Fig. 28.

the coordinates of P. We can find the area bounded by the arc OP, the axis of x , and the ordinate PN. (Fig. 28.) If this is z units of area we know that

$$\frac{dz}{dx} = y = x^2.$$

Hence
$$z = \frac{1}{3} x^3 + C,$$

where C is some number independent of x . This result holds for all values of x . But if we bring P along the curve to O we diminish x to zero and we also diminish z to zero. Hence C must be zero, and we have

$$z = \frac{1}{3} x^3.$$

70. Since the length of PN is x^2 units of length when the length of ON is x units of length, we have the result that the area bounded by the arc OP , the straight line ON , and the straight line PN is $\frac{1}{3}$ of the area of the rectangle $ONPL$. Hence the area bounded by the arc OP , the straight line PL and the straight line OL is $\frac{2}{3}$ of the area of the same rectangle. Hence also the area bounded by the arc QOP and the straight line PQ is $\frac{2}{3}$ of the area of the rectangle $PQMN$.

71. More generally, let AC and BD be two ordinates of a curve (Fig. 29), PN an intermediate ordinate, a the x -coordinate of A , b that of B , x , y the coordinates of P . Let the area

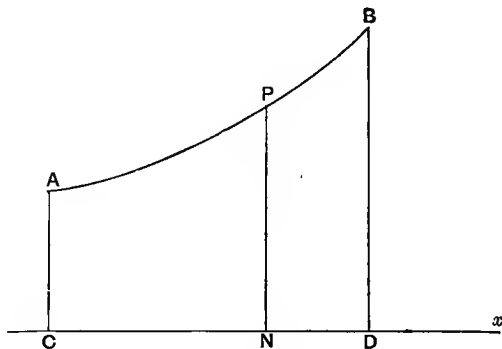


Fig. 29.

contained between the curve, the ordinates AC , BD , and the axis of x be S units of area, and let the area contained between the curve, the ordinates AC , PN , and the axis of x be z units of area. As in § 59 (a) we have

$$\frac{dz}{dx} = y.$$

Let $y = f(x)$ be the equation to the curve. Then we have

$$z = \int f(x) dx + C,$$

where C is independent of x . Let $\int f(x) dx = F(x)$, so that $F(x)$ is an integral of $f(x)$. Then $z = F(x) + C$. Now as P moves backwards along the curve towards A , z tends to zero, or we must have

$$F(a) + C = 0 \quad \text{or} \quad C = -F(a).$$

Hence
$$z = F(x) - F(a).$$

Also S is the value of z when $x = b$, and therefore

$$S = F(b) - F(a).$$

It is to be observed that we determine the curvilinear area $ACDB$ by regarding it as a particular value taken by the curvilinear area $ACNP$ as P moves along the curve from A .

72. In § 71 we showed how to determine the value of z when $x = b$ from the conditions :

$$(i) \quad \frac{dz}{dx} = f(x),$$

$$(ii) \quad z = 0 \quad \text{when} \quad x = a.$$

The solution is found by first finding an integral $F(x)$ of $f(x)$. The arbitrary constant, which may be added to the integral $F(x)$, or $\int f(x) dx$, is determined by the condition (ii). When this is done the function z is determined as $F(x) - F(a)$. There is nothing arbitrary in the expression for z . The required value is then found by substituting b for x . The relation of the required value of z to the indefinite integral, and to the two particular values a and b of x , is indicated by writing the required value in the form $\int_a^b f(x) dx$. This expression is called the "definite integral of $f(x)$ with respect to x between the lower limit a and the upper limit b ." It means the value when $x = b$ of a function which vanishes when $x = a$ and has its differential coefficient with respect to x equal to $f(x)$ for all values of x between a and b (inclusive).

By contradistinction any integral of $f(x)$, or $\int f(x) dx$, is often called an "indefinite integral" of $f(x)$.

EXAMPLES

Find the values of the following definite integrals (1)–(8):—

- (1) $\int_0^1 x dx$, (2) $\int_0^1 x^2 dx$, (3) $\int_0^2 (x^2 + x) dx$, (4) $\int_{-1}^1 x dx$,
 (5) $\int_{-1}^1 x^2 dx$, (6) $\int_0^1 (x^2 - x) dx$, (7) $\int_1^2 \frac{1}{x^2} dx$, (8) $\int_{0.01}^1 \frac{1}{x^2} dx$.

The results are

- (1) $\frac{1}{2}$, (2) $\frac{1}{3}$, (3) $\frac{14}{3}$, (4) 0,
 (5) $\frac{2}{3}$, (6) $-\frac{1}{6}$, (7) $\frac{1}{2}$, (8) 99.

73. The following example is important in connexion with the approximate evaluation of integrals¹.

Let P, R be two points of a parabola whose axis is parallel to the axis of y , PL, RN their ordinates, M the middle point of LN, MQ the ordinate of a

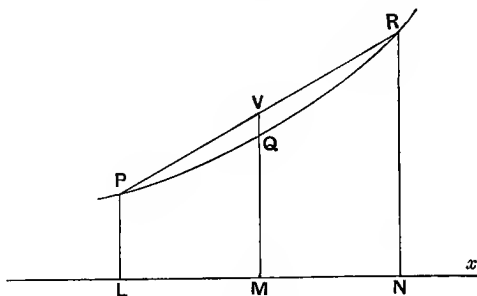


Fig. 30.

point Q on the parabola. (Fig. 30.) We may take the coordinates of the points P, Q, R to be: for P, $a - h$, y_1 ; for Q, a , y_2 ; for R, $a + h$, y_3 . If the equation to the parabola is

$$y = ax^2 + \beta x + \gamma,$$

¹ See Ch. X below.

and the number of units of area in the figure bounded by the arc PR, the ordinates PL, RN and the axis of x is S , we have

$$S = \int_{a-h}^{a+h} (ax^2 + \beta x + \gamma) dx.$$

Hence

$$\begin{aligned} S &= \frac{1}{3} a \{ (a+h)^3 - (a-h)^3 \} + \frac{1}{2} \beta \{ (a+h)^2 - (a-h)^2 \} + \gamma \{ (a+h) - (a-h) \} \\ &= \frac{1}{3} a (6a^2 h + 2h^3) + \frac{1}{2} \beta 4ah + \gamma 2h \\ &= 2h (aa^2 + \beta a + \gamma) + \frac{2}{3} ah^3. \end{aligned}$$

Now

$$aa^2 + \beta a + \gamma = y_2,$$

and

$$\begin{aligned} y_1 + y_3 - 2y_2 &= a \{ (a-h)^2 + (a+h)^2 - 2a^2 \} + \beta (a-h+a+h-2a) \\ &= 2ah^2. \end{aligned} \quad (\text{Cf. Ex. 1, p. 42.})$$

Therefore

$$\begin{aligned} S &= 2hy_2 + \frac{1}{3} h (y_1 + y_3 - 2y_2) \\ &= \frac{1}{3} h (y_1 + y_3 + 4y_2). \end{aligned}$$

74. Let the area of the trapezium PLNR be T units of area. Then

$$T = \frac{1}{2} 2h (y_1 + y_3) = h (y_1 + y_3),$$

and

$$T - S = \frac{2}{3} h (y_1 + y_3 - 2y_2) = \frac{2}{3} 2h \left\{ \frac{y_1 + y_3}{2} - y_2 \right\}.$$

Also the coordinates of V , the middle point of PR , are $a, \frac{y_1 + y_3}{2}$, and the length of VQ is $\frac{y_1 + y_3}{2} - y_2$ units of length. The area of the segment of the parabola bounded by the arc PQR and the chord PR is $T - S$ units of area. Hence the area of this segment is $\frac{2}{3}$ of that of a rectangle whose sides are LN and VQ .

The result in regard to the area of the parabolic segment is proved here for the case where a is positive or the curve is concave upwards. It is unaltered if the curve is concave downwards. In this case

$$S - T = \frac{2}{3} h (2y_2 - y_1 - y_3).$$

75. In general, when we wish to find the area of a figure, we may proceed as follows:—Let any straight line drawn across the figure parallel to the axis of y meet the boundary of the figure in two points P_1 and P_2 . Let y_1 and y_2 be the y -coordinates of these points, and let the suffix 1 be attached always to the upper point, the suffix 2 to the lower. Then $y_1 > y_2$. We write Y for $y_1 - y_2$. Let the straight line in question meet any straight line parallel to the axis of x in N . There will be two extreme positions of N , such as A, B in Fig. 31, and the whole figure will

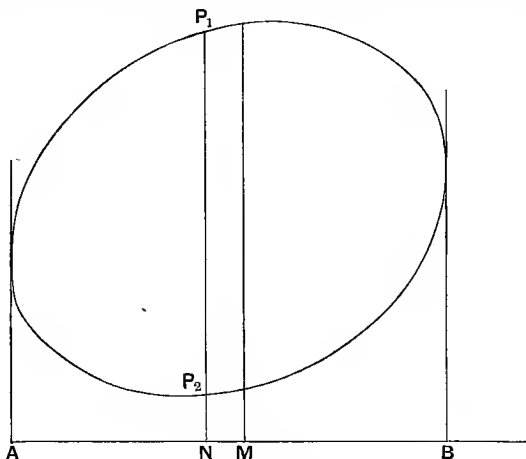


Fig. 31.

lie between two straight lines drawn through A and B parallel to the axis of y . Let a, x, b be the x -coordinates of A, N, B . We imagine N to move along the straight line AB from A to B . Then Y is a function of x , say $f(x)$.

When N moves to M , so that x becomes $x + \Delta x$, Y will become $Y + \Delta Y$, where ΔY may be positive or negative. The values taken by $f(x)$ when the upright straight line passes through a point between M and N need not lie between Y and $Y + \Delta Y$. The greatest of them may be a little greater than either Y or $Y + \Delta Y$

and the smallest of them a little less than either Y or $Y + \Delta Y$. Call the greatest of them K and the smallest k . As Δx tends to zero, K and k tend to the same limit, viz. Y .

Let the area of the part of the figure to the left of the upright line drawn through N be z units of area. When x is changed to $x + \Delta x$, z is changed to $z + \Delta z$, and the area of the strip of the figure between the upright lines drawn through N and M is Δz units of area. This area is less than that of a rectangle the lengths of whose sides are K and Δx units of length, but it is greater than that of a rectangle the lengths of whose sides are k and Δx units of length. Hence Δz lies between $K\Delta x$ and $k\Delta x$, and $\frac{\Delta z}{\Delta x}$ lies between K and k .

As Δx tends to zero, $\frac{\Delta z}{\Delta x}$ tends to a limit which is Y , and we have

$$\frac{dz}{dx} = Y = y_1 - y_2.$$

Let the area of the figure be S units of area. Then S is the value of z when $x = b$, and the value of z when $x = a$ is 0. Hence

$$S = \int_a^b (y_1 - y_2) dx.$$

It may be observed that this discussion applies to the area under a curve [§ 59 (a)]. In this case $y_2 = 0$, and y may be written for y_1 .

EXAMPLES

1. We may use the method of § 75 to find the area of a triangle. In each of the figures (Figs. 32, 33) we take the lengths of AB , OD , ON , P_1P_2 to be b , p , x , Y units of length, and we have $\frac{Y}{x} = \frac{b}{p}$. Hence

$$S = \int_0^p \frac{b}{p} x dx = \frac{1}{2} bp.$$

2. Prove that for a segment of a circle, of radius r units of length, cut off by a chord distant a units of length from the centre,

$$S = \int_a^r 2\sqrt{(r^2 - x^2)} dx.$$

[The indefinite integral will be evaluated in Ch. VIII.]

3. Draw roughly the graph of $y = x(1-x)$, and find the area contained between the curve and the axis of x . [Result, $\frac{1}{6}$ of a unit of area.]

4. Do the same for $y = x(1-2x)^2$. [Result, $\frac{1}{48}$ of a unit of area.]

5. Draw roughly the graph of $x(1-x)(2-x)$, and find the two areas enclosed by the curve and the axis of x . [Result, $\frac{1}{4}$ of a unit of area in both cases.]

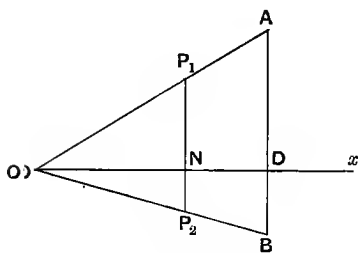


Fig. 32.

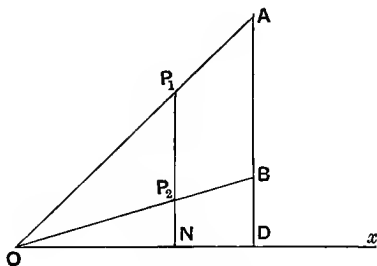


Fig. 33.

VOLUMES OF SOLIDS

76. We found in § 59 (b) that, if the distance between a plane drawn through the centre of a sphere and a parallel plane is x units of length, and the volume of the included solid is v units of volume,

$$\frac{dv}{dx} = \pi (a^2 - x^2),$$

where the radius of the sphere is a units of length. Also we have $v = 0$ when $x = 0$.

Hence
$$v = \pi \left(a^2 x - \frac{1}{3} x^3 \right).$$

If we put $x = a$, we have the volume of a hemisphere, viz.

$\frac{2}{3}\pi a^3$ units of volume. The volume of the complete sphere is $\frac{4}{3}\pi a^3$ units of volume.

77. The same reasoning will give us the volume of a cone. Let O be the vertex of the cone, let its height be a units of length, and the radius of its base b units of length. Take the axis of x along the axis of the cone. Through any point N on the axis let there pass a plane at right angles to the axis. The section of the

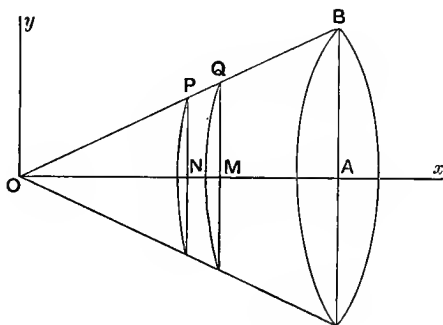


Fig. 34.

cone by the plane is a circle. Let PN be its radius, x, y the coordinates of P . The area of the circle is πy^2 units of area. Let the volume of the part of the cone included between this circle and the vertex be v units of volume. When x becomes $x + \Delta x$, v becomes $v + \Delta v$, and just as in § 59 (b) we find that Δv is intermediate between

$$\pi y^2 \Delta x \quad \text{and} \quad \pi (y + \Delta y)^2 \Delta x.$$

Hence

$$\frac{dv}{dx} = \pi y^2.$$

But we have

$$\frac{y}{x} = \frac{b}{a},$$

and therefore
$$\frac{dv}{dx} = \pi \frac{b^2}{a^2} x^2;$$

also $v = 0$ when $x = 0$. Hence

$$v = \frac{1}{3} \pi \frac{b^2}{a^2} x^3.$$

If the volume of the cone is V units of volume, V is the value of v when $x = a$, or we have

$$V = \frac{1}{3} \pi b^2 a.$$

The volume of the cone is $\frac{1}{3}$ of the volume of a cylinder whose height is the height of the cone and whose base is the base of the cone.

78. The same method may be used to find the volume of any solid of revolution. Let the curved bounding surface of the solid be generated by rotating a curve about an axis, which we take to be the axis of x . Let a fixed plane cut this axis at right angles, and let a plane passing through any point (x, y) on the curve also cut the axis of x at right angles. Let the volume of the portion of the solid contained between the two planes be v units of volume. Then v is a function of x , and $\frac{dv}{dx} = \pi y^2$.

79. This method can be generalized so as to apply to any solid. Let a fixed plane and a variable plane both cut the axis of x at right angles, and let x be the x -coordinate of the point where the variable plane cuts this axis; also let the area of the section of the solid by the variable plane be Z units of area, and the volume of the portion of the solid contained between the two planes be v units of volume. Then Z and v are functions of x and we have $\frac{dv}{dx} = Z$. If we know how to express Z as a function of x and to integrate this function we can find v .

80. As an example consider the volume of a pyramid on a triangular base. Let the area of the base be B units of area, and the length of the perpendicular OD let fall from the vertex O upon the plane of the base be p units of length. Take O as origin and the axis of x along OD . (Fig. 35.) Through any point N on the axis of x let there pass a plane at right angles to this axis cutting the edges OA , OB , OC of the pyramid in P , Q , R .

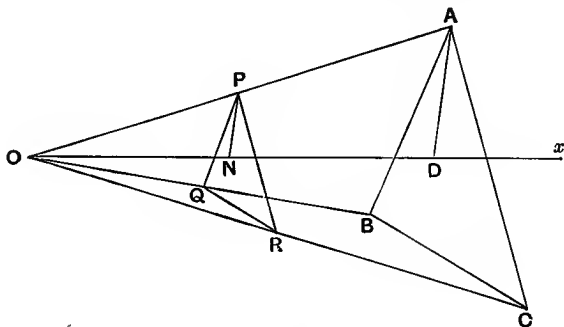


Fig. 35.

The area of the triangle PQR is what we have called Z units of area when the length of ON is x units of length. The volume of the pyramid with vertex O and base PQR is what we have called v units of volume; and we have the equation

$$\frac{dv}{dx} = Z.$$

Now the triangles PQR , ABC are similar, and the areas of similar figures are proportional to the areas of the squares described on corresponding sides. The ratio of a pair of corresponding sides is the same as that of any two corresponding lines in the two figures, e.g. the lines PN , AD or ON , OD . Hence we have

$$\frac{Z}{B} = \frac{x^2}{p^2},$$

and

$$\frac{dv}{dx} = \frac{B}{p^2} x^2;$$

also $v=0$ when $x=0$, and therefore

$$v = \frac{1}{3} \frac{B}{p^2} x^3.$$

If the volume of the pyramid is V units of volume, we find, by putting $x=p$, that

$$V = \frac{1}{3} Bp,$$

or the volume of the pyramid is $\frac{1}{3}$ of the volume of a prism of the same height and base.

Nothing in the argument depends upon the base having three sides, and the result holds for any pyramid.

EXAMPLES

1. Prove that the volume of a frustum of a pyramid or cone is

$$\frac{1}{3} \{A + a + \sqrt{Aa}\} h$$

units of volume, the areas of the two parallel faces being A and a units of area, and the distance between these faces being h units of length.

2. An arc of the parabola $y = \sqrt{x}$, terminated by the origin and a given point, revolves about the axis of x . Prove that the volume of the solid bounded by the surface traced out, and by the plane which passes through the given point and is at right angles to the axis of x , is one-half the volume of a cylinder of the same base and height.

3. An arc of the parabola $y = x^2$, terminated by the origin and a given point, revolves about the axis of x . Prove that the volume of the solid bounded as in Ex. 2 is one-fifth of the volume of a cylinder of the same base and height.

UNIFORMLY ACCELERATED MOTION

81. When a body, such as a falling stone, moves without rotation, and without change of size or shape, the motion of any point of it is the same as the motion of any other point of it. The coordinates of one point of the body at any instant specify the position of the body at that instant.

If every point of the moving body moves parallel to a fixed straight line we may speak of the body as moving in a straight line. If the straight line is the axis of x , the position of the body at any instant is specified by the x -coordinate of one point of it. If the unit of length is a foot, the point in question is

x feet to the right of the origin if x is positive, x feet to the left if x is negative. Let the position of the body be specified by x at the instant which is t seconds later than some chosen instant. If the point moves in the sense of increase of x the body has a velocity of $\frac{dx}{dt}$ feet per second in this sense. If the point moves

in the opposite sense the velocity is $-\frac{dx}{dt}$ feet per second. In both

cases $\frac{dx}{dt}$ is the measure in foot-second units of a certain quantity called the "velocity in the direction of increase of x ." If the

velocity is variable, the second differential coefficient $\frac{d^2x}{dt^2}$, if positive, measures, in foot-second units, the rate per second at which the velocity measured by $\frac{dx}{dt}$ increases. If $\frac{d^2x}{dt^2}$ is negative

$-\frac{d^2x}{dt^2}$ measures, in foot-second units, the rate per second at which

the velocity measured by $\frac{dx}{dt}$ diminishes. In both cases $\frac{d^2x}{dt^2}$ is the measure in foot-second units of a certain quantity called the "acceleration in the direction of increase of x ." (Cf. § 32.)

If the line of motion is the axis of y we have only to write y instead of x .

82. An unsupported body near the Earth's surface would, if the air offered no resistance, fall with a constant acceleration. This acceleration is called the "acceleration due to gravity." We generally write g for the numerical measure of this acceleration, meaning that this acceleration is g units of acceleration. In foot-second units g is 32.2.

We set up a system of coordinate axes so that the axis of y is the vertical at a place drawn upwards. Let the position of the body at the instant which is t seconds later than some chosen instant be specified by the coordinates x, y of one point of it. If

the body is free to move and we neglect the resistance of the air we have the equation

$$-\frac{d^2y}{dt^2} = g.$$

If the body is let fall at the instant from which t is reckoned, and if the value of y at that instant is y_0 , we have

$$y = y_0 \text{ and } \frac{dy}{dt} = 0 \text{ when } t = 0.$$

Write v for $\frac{dy}{dt}$. Then we have

$$-\frac{dv}{dt} = g$$

and $v = 0$ when $t = 0$.

$$\text{Hence } v = -gt.$$

$$\text{This is } \frac{dy}{dt} = -gt,$$

$$\text{which gives } y = -\frac{1}{2}gt^2 + C.$$

To make $y = y_0$ when $t = 0$ we must have

$$C = y_0.$$

$$\text{Hence } y = y_0 - \frac{1}{2}gt^2.$$

Any point of the body is $\frac{1}{2}gt^2$ feet lower down at the instant specified by t than it was at starting. The velocity of the body at this instant is gt feet per second in the downwards direction.

83. If the body instead of being let fall, is projected in some direction, we take the axis of x to be in the vertical plane passing through the direction of projection of one point of the body. Then every point of the body moves in a plane parallel to this plane. If, as before, we neglect the resistance of the air,

any point of the body has an acceleration of g units downwards and no horizontal acceleration. We have the equations

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g.$$

At starting the body has some horizontal velocity and some vertical velocity. We may suppose that

$$\frac{dx}{dt} = u \quad \text{and} \quad \frac{dy}{dt} = v, \quad \text{when } t = 0.$$

Then u and v are independent of t . The horizontal velocity at starting is u feet per second and the vertical velocity at starting is v feet per second, upwards.

We may take the origin at the point of projection, so that

$$x = 0 \quad \text{and} \quad y = 0 \quad \text{when } t = 0.$$

The equation $\frac{d^2x}{dt^2} = 0$ shows that $\frac{dx}{dt}$ is some constant; and the condition $\frac{dx}{dt} = u$ when $t = 0$ then shows that $\frac{dx}{dt} = u$ always. The horizontal velocity is always the same as it is at first. The equation $\frac{d^2y}{dt^2} = -g$, shows that $y = ut + C$ where C is constant, and the condition $x = 0$ when $t = 0$ shows that $C = 0$ or

$$x = ut.$$

The equation $\frac{d^2y}{dt^2} = -g$ shows that $\frac{dy}{dt} = -gt + A$, where A is constant, and the condition $\frac{dy}{dt} = v$ when $t = 0$ shows that $A = v$.

Hence $\frac{dy}{dt} = v - gt$. This equation shows that $y = vt - \frac{1}{2}gt^2 + B$, where B is constant, and the condition $y = 0$ when $t = 0$ shows that $B = 0$, or

$$y = vt - \frac{1}{2}gt^2.$$

We may eliminate t between the two equations $x = ut$ and $y = vt - \frac{1}{2}gt^2$. We get

$$y = -\frac{g}{2u^2}x^2 + \frac{v}{u}x.$$

The form of this equation shows (§ 35) that the path of the moving point is a parabola with its axis vertical, and that the curve lies below the tangent at the vertex (Fig. 36).

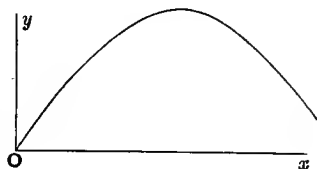


Fig. 36.

EXAMPLES

1. A body moves in a straight line with a constant acceleration f foot-second units. Prove that, if its velocity is u feet per second in the sense of increase of s at the instant from which t is reckoned, and v feet per second at the end of t seconds, $v = u + ft$. Prove that, if the body passes over s feet in t seconds, $s = ut + \frac{1}{2}ft^2$.

2. In the notation of Ex. 1, $\frac{dv}{dt} = f$ and $v = \frac{ds}{dt}$, so that $v \frac{dv}{dt} = f \frac{ds}{dt}$. Hence prove that $v^2 - u^2 = 2fs$.

3. If the motion of a body is such that the equation $\left(\frac{ds}{dt}\right)^2 - 2fs = \text{const.}$ is satisfied, f being a constant, prove that the body moves with a constant acceleration.

CHAPTER VI

LOGARITHMS AND THE EXPONENTIAL FUNCTION

84. WE have as a definition¹ of logarithms to base 10 (common logarithms) the statement that,

if $x = 10^y$,
 then $y = \log_{10} x$.

This definition would enable us to construct for ourselves a table of logarithms. We should begin by finding $\sqrt{10}$ by the ordinary arithmetical method correctly to a number of places of decimals. To 16 places we have

$$10^{\frac{1}{2}} = 3.1622776601683793.$$

From this we can find $10^{\frac{1}{4}}$ correctly to 8 places (1.77827941), then $10^{\frac{1}{8}}$ correctly to 4 places (1.3335), then $10^{\frac{1}{16}}$ correctly to 2 places (1.15). In the same way we can find by purely arithmetical methods the values of $10^{\frac{3}{16}}$, $10^{\frac{5}{16}}$, $10^{\frac{3}{8}}$, $10^{\frac{7}{16}}$, $10^{\frac{9}{16}}$, $10^{\frac{5}{8}}$, $10^{\frac{11}{16}}$, $10^{\frac{3}{4}}$, $10^{\frac{13}{16}}$, $10^{\frac{7}{8}}$, $10^{\frac{15}{16}}$. If then we take $x = 10^y$ we have the values of x for a number of values of y .

We can then put down the following table :

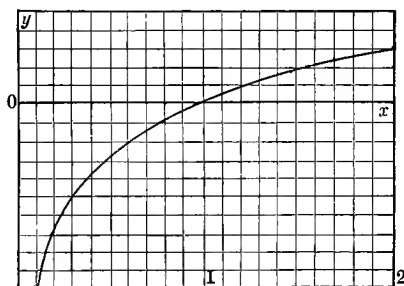
¹ A discussion of the definition will be found in Appendix III.

TABLE I.

y	0	$\frac{1}{18}$	$\frac{1}{8}$	$\frac{3}{18}$	$\frac{1}{4}$	$\frac{5}{18}$	$\frac{3}{8}$	$\frac{7}{18}$	$\frac{1}{2}$
x	1	1.15	1.33	1.54	1.78	2.05	2.37	2.74	3.16

y	$\frac{9}{18}$	$\frac{5}{8}$	$\frac{11}{18}$	$\frac{3}{4}$	$\frac{13}{18}$	$\frac{7}{8}$	$\frac{15}{18}$	1
x	3.65	4.22	4.87	5.62	6.49	7.50	8.66	10

In this table all the values of x are correct to 2 places of decimals. From this table we can draw a graph of $y = \log_{10} x$ between $x=1$ and $x=10$, and by means of the graph we can read off the logarithm of any number between 1 and 10. If the work is well done the value we can read off will certainly be correct to 1 place and often to 2. By means of the law of indices the table, or the graph, can be extended to values of x which do not lie between 0 and 1.



$$y = \log_{10} x$$

Fig. 37.

The graph of $\log_{10} x$ between $x=0.1$ and $x=2$ is shown in Fig. 37.

We see that anyone who took trouble enough, and made no

mistakes, could construct a table of logarithms, correct to as many decimal places as he might wish, by merely finding square roots, just as a business man might make his own ready reckoner. But it saves time to buy one. For a similar reason we buy a Table of Logarithms¹.

85. In working with logarithms the most important formulæ are

$$\log(ab) = \log a + \log b,$$

$$\log\left(\frac{a}{b}\right) = \log a - \log b,$$

$$\log(a^n) = n \log a.$$

These hold for any base. In regard to change of the base the chief formulæ are

$$\log_a x = (\log_b x) (\log_a b),$$

$$(\log_a b) \times (\log_b a) = 1.$$

86. We consider $\log_{10} x$ as a function of x , and seek to differentiate it. We have

$$\log_{10}(x+h) - \log_{10} x = \log_{10} \frac{x+h}{x} = \log_{10} \left(1 + \frac{h}{x}\right),$$

and

$$\begin{aligned} \frac{\log_{10}(x+h) - \log_{10} x}{h} &= \frac{1}{h} \log_{10} \left(1 + \frac{h}{x}\right) = \frac{1}{x} \cdot \frac{x}{h} \cdot \log_{10} \left(1 + \frac{h}{x}\right) \\ &= \frac{1}{x} \log_{10} \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}. \end{aligned}$$

We wish to show that this expression tends to a limit when h tends to zero. We write n for $\frac{x}{h}$, and have

$$\frac{\log_{10}(x+h) - \log_{10} x}{h} = \frac{1}{x} \log_{10} \left(1 + \frac{1}{n}\right)^n.$$

Now it is not difficult to convince ourselves that $\left(1 + \frac{1}{n}\right)^n$

¹ The tables of logarithms in books of tables are not constructed by finding square roots in the manner explained above, but by a different method depending upon the use of infinite series.

tends to a limit as n increases. We consider $\log_{10} \left(1 + \frac{1}{n}\right)^n$, or $n \log_{10} \left(1 + \frac{1}{n}\right)$. In Table II. we have the values which this expression takes as n increases from 1 to 9. We see that each value is a little greater than the one before it, but the differences get smaller.

TABLE II.

n	1	2	3	4	5	6	7	8	9
$n \log_{10} \left(1 + \frac{1}{n}\right)$	0.3010	0.3522	0.3748	0.3876	0.3959	0.4017	0.4059	0.4092	0.4118

Now we go a little faster, and put down in Table III. the values which the expression takes when n is equal to 10, 20, ... 90. We see that the logarithm always increases but the differences are getting much smaller.

TABLE III.

n	10	20	30	40	50	60	70	80	90
$n \log_{10} \left(1 + \frac{1}{n}\right)$	0.4139	0.4238	0.4272	0.4290	0.4300	0.4307	0.4312	0.4316	0.4319

Again we will go faster still and put down in Table IV. the values which the expression takes when n is equal to 100, 200, ... 900. We see that the logarithm tends to a constant value.

TABLE IV.

n	100	200	300	400	500	600	700	800	900
$n \log_{10} \left(1 + \frac{1}{n}\right)$	0.4321	0.4332	0.4336	0.4338	0.4339	0.4339	0.4340	0.4340	0.4341

If now we go further, and take n to be 1000, 2000, ... we find that all the values of the logarithm, correct to 3 places, are 0.434. With a seven-figure table we cannot be sure of more than three places. Hence we conclude that, as n increases, $\log_{10} \left(1 + \frac{1}{n}\right)^n$ probably tends to a limit, which is approximately equal to 0.434, and that $\left(1 + \frac{1}{n}\right)^n$ probably tends to a limit which is approximately equal to 2.72.

It can be proved formally that $\left(1 + \frac{1}{n}\right)^n$ tends to a limit as n increases¹. This limit is a perfectly definite number denoted by e . The value of e correct to 4 places of decimals is 2.7183, and the value of $\log_{10} e$ correct to 4 places of decimals is 0.4343. We shall denote the number $\log_{10} e$ by M .

Now we have the result

$$\frac{d \log_{10} x}{dx} = \frac{M}{x}.$$

87. Instead of taking logarithms to base 10 we might take logarithms to any base a . We should have as a definition the statement that if $x = a^y$ then $y = \log_a x$. By the process adopted in § 86 we should find

$$\frac{\log_a(x+h) - \log_a(x)}{h} = \frac{1}{x} \log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}},$$

and thence

$$\frac{d \log_a x}{dx} = \frac{1}{x} \log_a e = \frac{1}{x \log_e a}.$$

In particular we might take e as base, and find

$$\frac{d \log_e x}{dx} = \frac{1}{x}.$$

¹ See Appendix IV.

88. This result gives us a new standard form of integral, viz.

$$\int \frac{1}{x} dx = \log_e x. \quad (\text{B})$$

The result may also be written

$$\int \frac{1}{x} dx = \frac{1}{M} \log_{10} x,$$

where

$$\frac{1}{M} = \log_e 10 \doteq 2.3026.$$

We can write down also by the method of §§ 66, 68 the more general results

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \log_e (ax+b),$$

$$\int \frac{1}{z} \frac{dz}{dx} dx = \log_e z.$$

THE EXPONENTIAL FUNCTION

89. When $y = \log_e x$, $x = e^y$, and we know that

$$\frac{dy}{dx} = \frac{1}{x}.$$

Hence

$$\frac{dx}{dy} = x,$$

or

$$\frac{d(e^y)}{dy} = e^y.$$

Here e^y is regarded as a function of y . It is known as the "exponential function." If we write x in place of y , e^x is the exponential function of x , and we have

$$\frac{d(e^x)}{dx} = e^x.$$

In the following Table the values of e^x are given, correctly to 4 places of decimals, for a number of values of x lying between -1 and 1 . Values of e^x in other ranges of values of x can be deduced by help of the law of indices.

x	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
e^x	0.3679	0.4066	0.4493	0.4966	0.5488	0.6065	0.6703	0.7408	0.8187	0.9048	1

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
e^x	1.1052	1.2214	1.3499	1.4918	1.6487	1.8221	2.0138	2.2255	2.4596	2.7183

The graph of e^x between $x = -1$ and $x = 1$ is shown in Fig. 38.

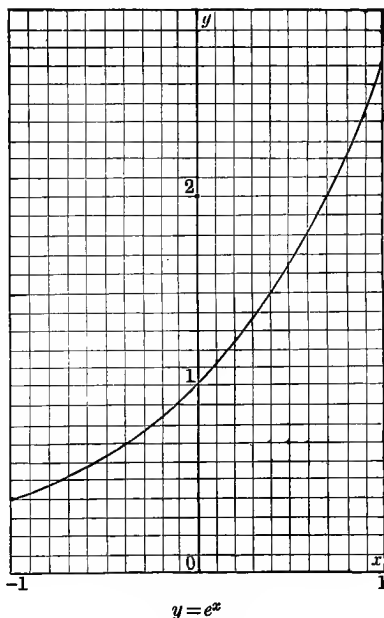


Fig. 38.

90. If x is negative and numerically large e^x is positive and small, and it diminishes rapidly as the negative value of x increases numerically. In other words, as the positive number x increases e^{-x} continually diminishes and rapidly assumes very small values. On the other hand, if the index x is positive, e^x continually increases with x , and rapidly assumes very large values.

91. If we apply the rule for differentiating a function of a function (§ 25) we find at once that

$$\frac{d(e^{ax})}{dx} = ae^{ax},$$

a being any constant. This result gives us an important integral, viz.,

$$\int e^{ax} dx = \frac{1}{a} e^{ax}. \quad (C)$$

In this formula a may be any positive or negative number.

92. The function a^x is often called an exponential function. Since

$$a = e^{\log_e a},$$

we have

$$a^x = e^{x \log_e a},$$

and

$$\frac{d(a^x)}{dx} = \log_e a \cdot e^{x \log_e a} = \log_e a \cdot a^x.$$

This gives us an important limit. If we attempted to differentiate a^x directly we should form the quotient

$$\frac{a^{x+h} - a^x}{h},$$

which is the same as

$$a^x \frac{a^h - 1}{h}.$$

It follows that, as h tends to zero, $\frac{a^h - 1}{h}$ tends to a limit which is $\log_e a$.

In particular $\frac{10^h - 1}{h}$ tends to the limit $\frac{1}{M}$, approximately equal to 2.3026 ;

$\frac{1^h - 1}{h}$ tends to the limit zero, and $\frac{e^h - 1}{h}$ tends to the limit 1. The number e is distinguished from all other numbers by the fact that when $a = e$ the limit of $\frac{a^h - 1}{h}$ is 1 ; when $a > e$ or $a < e$ the limit is not 1.

EXAMPLES

1. Differentiate the following (1)–(12) :—

- | | | |
|--------------------------------|-----------------------|--------------------------------|
| (1) $\log_e (2+x)$, | (2) $\log_e (1-2x)$, | (3) $\log_e \frac{1+x}{1-x}$, |
| (4) $\log_e \frac{x+1}{x-1}$, | (5) $x \log_e x$, | (6) $\{\log_e (x)\}^2$, |
| (7) xe^x , | (8) $x^2 e^x$, | (9) e^{-x} , |
| (10) e^{-x^2} , | (11) $(1-x)e^x$, | (12) $(1-x)^2 e^x$. |

2. Prove that $\frac{d[\log_e \{x + \sqrt{(x^2 + C)}\}]}{dx} = \frac{1}{\sqrt{(x^2 + C)}}$,

C being independent of x . Cf. Ex. (ii), p. 32.

3. Prove that if h is very small $\log_e (1+h) \doteq h$, and

$$\log_e (x+h) - \log_e x \doteq \frac{h}{x}.$$

4. Prove that when h is very small $e^h \doteq 1+h$.

5. Integrate the following (1)–(6) :—

- | | | | |
|-----------------------|----------------------------|-----------------------|----------------|
| (1) $\frac{1}{1+x}$, | (2) $\frac{1}{1-x}$, | (3) $\frac{1}{x-1}$, | (4) e^{-x} , |
| (5) xe^{-x^2} , | (6) $\frac{\log_e x}{x}$. | | |

Verify the results by differentiation.

6. Integrate $\frac{1}{\sqrt{(x^2+1)}}$ and $\frac{1}{\sqrt{(x^2-1)}}$.

7. Prove that, if y is a function of x , $\frac{d(ye^{ax})}{dx} = e^{ax} \left(\frac{dy}{dx} + ay \right)$.

8. Prove that, if m is constant, the equation

$$\frac{d^2y}{dt^2} = m^2y$$

is satisfied by putting $y = Ae^{mt} + Be^{-mt}$, where A and B are any constants.

9. Prove that $\frac{d^2y}{dx^2} + a \frac{dy}{dx} = e^{-\frac{1}{2}ax} \left\{ \frac{d^2(e^{\frac{1}{2}ax}y)}{dx^2} - \frac{1}{4}a^2 e^{\frac{1}{2}ax}y \right\}$,

a being a constant and y a function of x . Hence prove that, if x satisfies the equation $\frac{d^2x}{dt^2} + k \frac{dx}{dt} + n^2x = 0$, $xe^{\frac{1}{2}kt}$ satisfies the equation

$$\frac{d^2(xe^{\frac{1}{2}kt})}{dt^2} + \left(n^2 - \frac{1}{4}k^2 \right) (xe^{\frac{1}{2}kt}) = 0,$$

k and n being constants, and x a function of t .

APPLICATIONS OF THE EXPONENTIAL FUNCTION

93. We consider some problems which can be solved by means of the exponential function.

In Physical Chemistry¹ we meet frequently with problems in which a number x , representing the measure of concentration of a solution, is to be determined in terms of a number t , representing the measure of the time elapsed since some instant, by an equation of the form $\frac{dx}{dt} = f(x)$, where $f(x)$ is a given function.

For example, the rate of inversion of cane sugar at any instant during the process of inversion is proportional to the amount that has not been inverted at that instant. The amount is specified by the degree of concentration. If a measures the initial concentration, x the concentration of invert sugar at the end of t minutes, we have the equation

$$\frac{dx}{dt} = k(a - x),$$

where k is a constant, which is found to have the value 0.0015 approximately. We have to solve this equation, and we must also satisfy the condition that $x = 0$ when $t = 0$.

We have
$$\frac{dt}{dx} = \frac{1}{k(a - x)},$$

or
$$\frac{d(kt)}{dx} = \frac{1}{a - x},$$

and therefore
$$kt = \int \frac{1}{a - x} dx + C$$

$$= -\log_e (a - x) + C.$$

To make $x = 0$ when $t = 0$ we must have

$$C = \log_e a,$$

and therefore
$$kt = \log_e a - \log_e (a - x) = \log_e \frac{a}{a - x},$$

¹ See J. W. Mellor, *Chemical Statics and Dynamics*, London, 1894.

or
$$\frac{a}{a-x} = e^{kt},$$

or
$$\frac{a-x}{a} = e^{-kt},$$

so that
$$\frac{x}{a} = 1 - e^{-kt}.$$

This formula gives the solution of the problem.

94. As another example, let the velocity of chemical change be given by an equation of the form

$$\frac{dx}{dt} = k(a-x)(b-x),$$

where k , a , b are constants, and let $x=0$ when $t=0$. We have to express x in terms of t . We shall suppose that b is the greater of the two numbers a and b .

We have
$$k \frac{dt}{dx} = \frac{1}{(a-x)(b-x)},$$

and we observe that we know how to integrate the sum $\frac{1}{a-x} + \frac{1}{b-x}$ and also how to integrate the difference $\frac{1}{a-x} - \frac{1}{b-x}$, and further we observe that

$$\frac{1}{a-x} - \frac{1}{b-x} = \frac{b-a}{(a-x)(b-x)}.$$

Multiplying both sides of our equation by $(b-a)$, we have

$$k(b-a) \frac{dt}{dx} = \frac{1}{a-x} - \frac{1}{b-x}.$$

Hence
$$\begin{aligned} k(b-a)t &= \int \left(\frac{1}{a-x} - \frac{1}{b-x} \right) dx + C \\ &= -\log_e(a-x) + \log_e(b-x) + C \\ &= \log_e \frac{b-x}{a-x} + C. \end{aligned}$$

To make $x=0$ when $t=0$ we must have

$$C = -\log_e \frac{b}{a} = \log_e \frac{a}{b},$$

and therefore
$$k(b-a)t = \log_e \left(\frac{b-x}{a-x} \cdot \frac{a}{b} \right),$$

or
$$\frac{a(b-x)}{b(a-x)} = e^{k(b-a)t},$$

so that
$$x(b e^{k(b-a)t} - a) = ab(e^{k(b-a)t} - 1),$$

and the solution of the problem is given by the equation

$$x = ab \frac{e^{k(b-a)t} - 1}{b e^{k(b-a)t} - a}.$$

95. In the theory of Electric Currents¹ we meet frequently with problems in which a number i , representing the measure of a current, is to be determined in terms of a number t , representing the measure of an interval of time elapsed since some instant, by an equation of the form

$$L \frac{di}{dt} + Ri = E,$$

where L and R are constants, representing the measures of the coefficient of self-induction and the resistance of a circuit, and E is a constant or a function of the time, representing the measure of an impressed electromotive force. We have to determine i so as to satisfy this equation and the condition that $i = 0$ when $t = 0$.

We take the case where E is a constant, and write

$$b = \frac{R}{L}.$$

Then
$$\frac{di}{dt} + bi = \frac{E}{L}.$$

Now (Ex. 7, p. 92)

$$\frac{d(i e^{bt})}{dt} = e^{bt} \left(\frac{di}{dt} + bi \right),$$

and therefore
$$\frac{d(i e^{bt})}{dt} = \frac{E}{L} e^{bt},$$

so that
$$i e^{bt} = \frac{E}{Lb} e^{bt} + C = \frac{E}{R} e^{bt} + C.$$

¹ See J. J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism*, Ch. xi, Cambridge, 1895.

To make $i = 0$ when $t = 0$ we must have

$$C = -\frac{E}{R},$$

and therefore

$$ie^{bt} = \frac{E}{R}(e^{bt} - 1),$$

or

$$i = \frac{E}{R}(1 - e^{-bt}).$$

The solution of the problem is given by the equation

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right).$$

This formula gives the intensity of the current produced in a circuit by a constant electromotive force, the circuit being closed at the instant specified by $t = 0$.

96. We may take account of the resistance of the air to the motion of a falling body by assuming that this resistance is proportional to the velocity. This assumption gives a good approximation to the effect of the air on the motion of a small body which is not moving very fast.

Let the body fall through s feet in t seconds from the start. Its velocity at the instant specified by t is $\frac{ds}{dt}$ feet per second, and its acceleration in the downwards direction is $\frac{d^2s}{dt^2}$ foot-second units of acceleration. The force of the earth's gravity gives it an acceleration g , or 32.2 , foot-second units, in the downwards direction. The resistance of the air gives it an acceleration $k \frac{ds}{dt}$ foot-second units in the upwards direction, k being a constant. We have therefore the equation

$$\frac{d^2s}{dt^2} = g - k \frac{ds}{dt},$$

and the conditions that $s = 0$ and $\frac{ds}{dt} = 0$ when $t = 0$. The above equation is

$$\frac{d^2s}{dt^2} + k \frac{ds}{dt} = g.$$

As in § 95 we may write this equation in the form

$$\frac{d \left(e^{kt} \frac{ds}{dt} \right)}{dt} = g e^{kt},$$

so that
$$e^{kt} \frac{ds}{dt} = \int g e^{kt} dt + C = \frac{g}{k} e^{kt} + C.$$

To make $\frac{ds}{dt} = 0$ when $t = 0$ we must have

$$C = -\frac{g}{k},$$

and therefore

$$\frac{ds}{dt} = \frac{g}{k} (1 - e^{-kt}).$$

We see that, as t increases, $\frac{ds}{dt}$ increases, but that it never becomes greater than $\frac{g}{k}$. The velocity tends to a limiting velocity $\frac{g}{k}$ feet per second. This velocity is called the "terminal velocity."

To find s in terms of t we have the equation

$$\frac{ds}{dt} = \frac{g}{k} (1 - e^{-kt}),$$

and the condition that $s = 0$ when $t = 0$. The equation gives

$$s = \frac{g}{k} t + \frac{g}{k^2} e^{-kt} + C',$$

where C' is a constant. To make $s = 0$ when $t = 0$ we must have

$$C' = -\frac{g}{k^2}.$$

Hence

$$s = \frac{g}{k} t - \frac{g}{k^2} (1 - e^{-kt}).$$

We see that if the body falls for a long time (so that t is large), the distance through which it falls is nearly the same as if it fell with the uniform velocity $\frac{g}{k}$ feet per second for $t - \frac{1}{k}$ seconds.

EXAMPLES

1. In the problem of § 93, taking $a=10.023$, $x=0$ when $t=0$, $x=1.946$ when $t=60$, find k , and make a table of the values of x when $t=30, 90, 120, 150, 180$.

2. In the problem of § 94, taking $a=10.023$, $b=89.977$, $x=0$ when $t=0$, $x=1.946$ when $t=60$, find k , and make a table of the values of x when $t=30, 90, 120, 150, 180$.

3. In the problem of § 95, taking $E=20$, $R=2$, $L=4$, find the value of i when $t=1, 10, 60$. Do the same, taking $E=20$, $R=4$, $L=2$.

4. A tank is being emptied, in such a way that the rate at which the water is flowing out at any instant is proportional to the amount left in at that instant. If half the water flows out in five minutes, how much will flow out in 10, 15, 20 minutes?

5. Rain is falling with a velocity of 60 feet per second. Assuming that this differs very little from the terminal velocity (§ 96), and that the resistance is proportional to the velocity, find approximately the retardation of the drops by the resistance, and prove that the time taken to acquire a velocity of 54 feet per second is 4.29 seconds nearly, also that the distance through which a drop falls in the first second of its fall is 14 feet nearly.

CHAPTER VII

TRIGONOMETRIC FUNCTIONS

97. If the radius of a circle is r units of length, and the length of any arc is s units of length, the number $\frac{s}{r}$ is a measure of the angle which the arc subtends at the centre. If we put θ for $\frac{s}{r}$, the angle is θ radians¹. A right angle is $\frac{1}{2}\pi$ radians, or 1.5708 radians approximately, and a radian is $\frac{2}{\pi}$ right angles or 0.6366 right angles approximately. The area of the sector which stands on an arc is $\frac{1}{2}r^2\theta$ units of area when the angle subtended by the arc at the centre is θ radians, the radius being r units of length.

98. It is convenient to think of angles as capable of taking any magnitude, greater or less than two right angles. For this purpose we think of the angle as traced out by one of its sides revolving about the vertex of the angle. We may think of the revolving line as starting from the position occupied by one side of the angle, and turning round the vertex, until it comes into the position occupied by the other side.

¹ A discussion of the radian measure of angles will be found in *Appendix V*.

We generally think of the vertex of the angle as the origin of a system of coordinates x, y , and of the starting position of the revolving line as the axis of x drawn towards the right from the origin. We also agree that, when θ is a positive number, an angle of θ radians is traced out by a line revolving in the opposite direction to the hands of a watch, placed face upwards on the paper (Fig. 39). According to this convention the sides of an angle of $2n\pi + \theta$ radians (n being a positive integer) are in the same positions as the sides of an angle of θ radians.

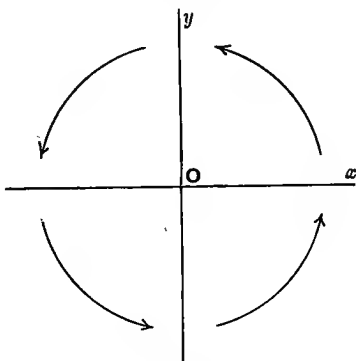


Fig. 39.

99. Angles of magnitude greater than two right angles are important in connexion with the rotation of bodies. If we think, for instance, of a rotating fly-wheel, and of the motion of a line traced on one of its faces, we see that the line turns through 2π radians in every complete revolution of the wheel. In any interval, say t seconds, the line will turn through some angle, say θ radians, and θ may be greater or less than π according to circumstances. As the wheel turns, θ increases, and θ is some function of t . If the wheel turns uniformly, θ is a simple multiple of t , and $\frac{\theta}{t}$ is the measure in radians per second of the "angular velocity" of the wheel. Whether the wheel turns uniformly or not, $\frac{d\theta}{dt}$ is the measure in radians per second of the angular velocity.

100. Let P be any point in the plane of the coordinate axes, and let OP be one side of an angle of which the other side is Ox . Let the measure of the angle traced out by the revolving line as it turns from the position Ox to the position OP be θ radians, and let the distance OP be r units of length. Let x, y be the coordinates of P . Then the numbers $\frac{x}{r}, \frac{y}{r}$ depend upon θ , but not on anything else. They are functions of θ , called the *cosine* and the *sine*. We write

$$\frac{y}{r} = \sin \theta, \quad \frac{x}{r} = \cos \theta.$$

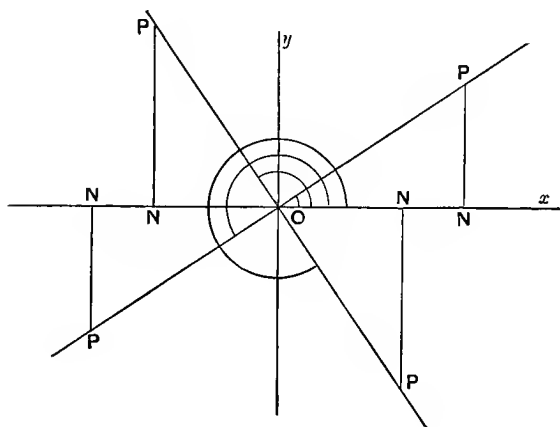


Fig. 40.

101. Since $x^2 + y^2 = r^2$, we have the result which is usually written

$$\sin^2 \theta + \cos^2 \theta = 1,$$

in which $\sin^2 \theta$ means $(\sin \theta)^2$, and $\cos^2 \theta$ means $(\cos \theta)^2$.

102. In Fig. 40 the two lines marked POP are at right angles to each other, and all the triangles marked PON are equal

in all respects. Hence, attending to the signs of x, y we have

$$\cos \theta = \sin \left(\frac{1}{2} \pi + \theta \right), \quad \sin \theta = -\cos \left(\frac{1}{2} \pi + \theta \right) \dots (1).$$

Further

$$\sin \theta = -\sin (\pi + \theta), \quad \cos \theta = -\cos (\pi + \theta) \dots \dots (2),$$

and $\sin \theta = \sin (2\pi + \theta), \quad \cos \theta = \cos (2\pi + \theta) \dots \dots (3).$

103. In Fig. 41 the two lines marked OP, OP' are equally inclined to the axis of x . If the acute angle xOP is θ radians the obtuse angle xOP' is $\pi - \theta$ radians. Attending to the signs of x, y we see that

$$\sin (\pi - \theta) = \sin \theta, \quad \cos (\pi - \theta) = -\cos \theta \dots \dots (4).$$

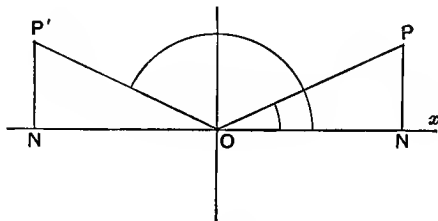


Fig. 41.

By means of these formulæ we can write down the values of $\sin \theta$ and $\cos \theta$ for any positive value of θ when the value of $\sin \theta$ is known for values of θ between 0 and $\frac{1}{2} \pi$.

104. It is sometimes important to observe that, any two numbers a and b being given, it is possible to find one, and only one, positive number r and, at the same time, one, and only one, number θ , between 0 and 2π , so as to make

$$a = r \cos \theta, \quad b = r \sin \theta.$$

Either or both of the numbers a and b may be negative; but, whether they are negative or positive, the two are coordinates of one, and only one, point, say P . This point is at a perfectly definite distance from the origin, and the numerical measure of this distance in terms of the unit of length is the

required number r . The angle through which a straight line, starting from the position Ox , and revolving through less than one complete revolution in the counter-clockwise sense, would have to turn in order to come into the position OP is a perfectly definite angle, and the numerical measure of this angle in radians is the required number θ .

105. We might make our own table of sines, just as we could make our own table of logarithms. To see how this could be done we notice that if we draw an angle, *e.g.* $\frac{1}{4}$ of a right angle as accurately as possible, we can mark a point P on one of its sides, draw a perpendicular from P to the other side, and measure as accurately as possible the length of the perpendicular and the distance of the point P from the vertex of the angle. Then the fraction

$$\frac{\text{number of units of length in the length of the perpendicular}}{\text{number of units of length in the distance}}$$

is the sine of the angle, and the sine can therefore be determined with a degree of accuracy that depends only on the accuracy of our measurement. Just as in the case of logarithms, we do not actually calculate a table of sines, but buy one¹.

In an ordinary table of sines the angles are given in degrees and minutes (a minute is $\frac{1}{60}$ of a degree, and a degree is $\frac{1}{90}$ of a right angle). We can reduce the measure of the angle to radians by the rule that 180 degrees is π radians. By means of the ordinary tables we could construct a new table² in which the angle is expressed in radians, and the sine of each angle is given correctly to a few places of decimals. The following table gives a few corresponding values correctly to three places.

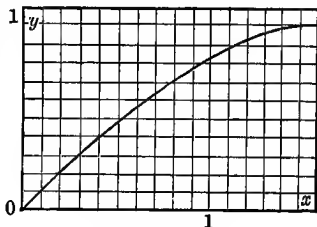
¹ The tables of sines in books of tables are not constructed by measuring lengths in the manner described, but by a different method depending upon the use of infinite series.

² Tables of this kind have been constructed by C. Burrau. See his 'Tables of cosine and sine of real and imaginary angles expressed in radians.' Berlin, 1907.

θ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$\sin \theta$	0	0.100	0.199	0.296	0.389	0.479	0.565	0.644	0.717

θ	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	
$\sin \theta$	0.783	0.841	0.891	0.932	0.964	0.985	0.997	1	

106. In what precedes θ is a variable number and $\sin \theta$ is a function of that variable. We thought of θ as the number of radians in the measure of some angle, but we need not think of it in that way. We may take the two numbers θ and $\sin \theta$ to be simultaneous values of two variable numbers x and y , and then y is a certain function of x , the sine of x . We may use the table in § 105 to plot the graph of $y = \sin x$ when x lies between 0 and $\frac{1}{2}\pi$ (or 1.5708). The graph is shown in Fig. 42.



$y = \sin x$.

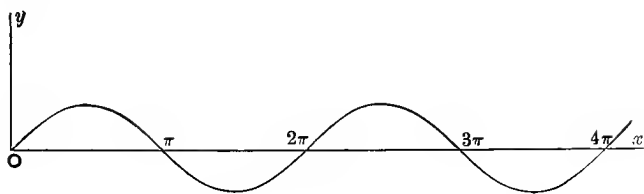
Fig. 42.

We may now use the results of §§ 102, 103 to continue the graph of $\sin x$ beyond

$x = \frac{1}{2}\pi$. The graph consists of a succession of exactly equal and similar bays lying alternately above and below the axis of x , as shown in Fig. 43.

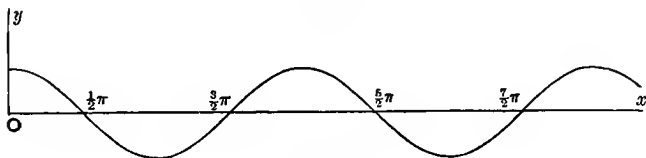
We may then use the result $\cos x = \sin\left(\frac{1}{2}\pi + x\right)$ to draw the graph of $y = \cos x$ for positive values of x . The result is shown in Fig. 44 which is obtained by shifting the curve in Fig. 43 through $\frac{1}{2}\pi$ units of length towards the left hand.

107. To define $\sin x$ and $\cos x$ for negative values of x , we simply make it a rule that the whole of the graph in each case



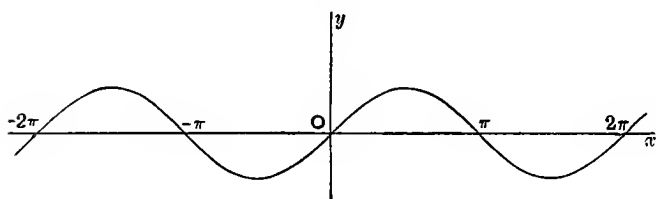
$$y = \sin x.$$

Fig. 43.



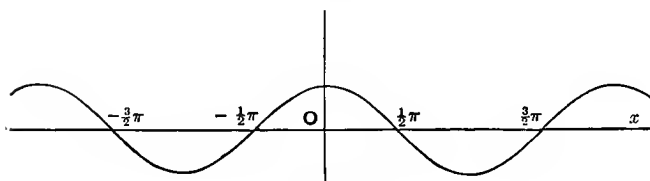
$$y = \cos x.$$

Fig. 44.



$$y = \sin x.$$

Fig. 45.



$$y = \cos x.$$

Fig. 46.

consists of an endless succession of exactly like bays alternately above and below the axis of x , Figs. 45, 46. Now if x is negative, $-x$ is positive. We put x' for $-x$. Then as x diminishes (algebraically) from zero, x' increases above zero. Now the graph shows that the values of $\sin x$ and $\sin x'$ are always equal in magnitude but opposite in sign. Hence the definition gives the equation

$$\sin x' = -\sin x, \text{ or } \sin(-x) = -\sin x.$$

Again the graph of the cosine shows that the values of $\cos x$ and $\cos x'$ are always equal and have the same sign, and the definition therefore gives the equation

$$\cos x' = \cos x, \text{ or } \cos(-x) = \cos x.$$

108. The other trigonometric functions are defined by means of the sine and cosine. We write down the defining equations

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \operatorname{cosec} x = \frac{1}{\sin x} \dots (1).$$

With a view to drawing the graph of $\tan x$ we observe that, when $x = \frac{1}{2}\pi$, $\sin x = 1$, and, when $x = \frac{3}{2}\pi$, $\sin x = -1$. Generally

$\sin x$ is either 1 or -1 whenever x is an odd multiple of $\frac{1}{2}\pi$.

These are the greatest and least values of $\sin x$. In like manner $\cos x = 1$ when $x = 0$ or x is an even multiple of π , and $\cos x = -1$ when x is an odd multiple of π . These are the greatest and least values of $\cos x$. Whenever $\sin x$ is 1 or -1 , $\cos x = 0$, and therefore $\tan x$ is not defined by the above equation when x is an odd multiple of $\frac{1}{2}\pi$, for we cannot divide 1 or -1 by 0. When x is

nearly equal to $\frac{1}{2}\pi$, $\sin x$ is nearly equal to 1 and $\cos x$ is a very small number, positive if x is a little less than $\frac{1}{2}\pi$, negative if x is a little greater than $\frac{1}{2}\pi$. Hence $\tan x$ is a very great positive

number when x is a little less than $\frac{1}{2}\pi$, and $-\tan x$ is a very great positive number when x is a little greater than $\frac{1}{2}\pi$. By the results of §§ 102, 107 we have

$$\tan\left(\frac{1}{2}\pi + x\right) = -\frac{1}{\tan x}, \quad \tan(\pi + x) = \tan x, \quad \tan(-x) = -\tan x \dots (2),$$

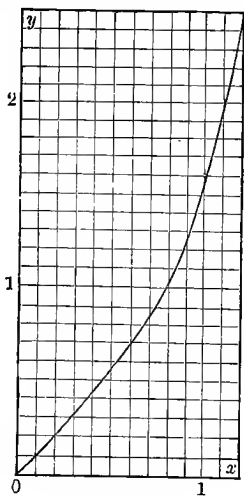
and we therefore know the value of $\tan x$ for any value of x except odd multiples of $\frac{1}{2}\pi$ if we know the values of $\tan x$ between 0 and $\frac{1}{2}\pi$ (0 included, $\frac{1}{2}\pi$ excluded). Just as in § 105 we may form a table of the values of $\tan x$ that correspond to values of x between 0 and $\frac{1}{2}\pi$. In the table given on p. 108 the values of $\tan x$ are correct to 3 places of decimals.

The graph of $\tan x$ for values of x between 0 and 1.2 is shown in Fig. 47. For values of x between

0 and $-\frac{1}{2}\pi$ the graph is obtained by applying the rule $\tan(-x) = -\tan x$. The more complete graph (Fig. 48) is obtained by repeating the same curve in those strips of the plane which are bounded by pairs of consecutive lines given by equations of the form

$$x = \text{an odd multiple of } \frac{1}{2}\pi,$$

in the same way as the complete graph of $\sin x$ is obtained by repeating the same curve in every strip of the plane that is bounded by a pair of consecutive lines along which x is a multiple of 2π .

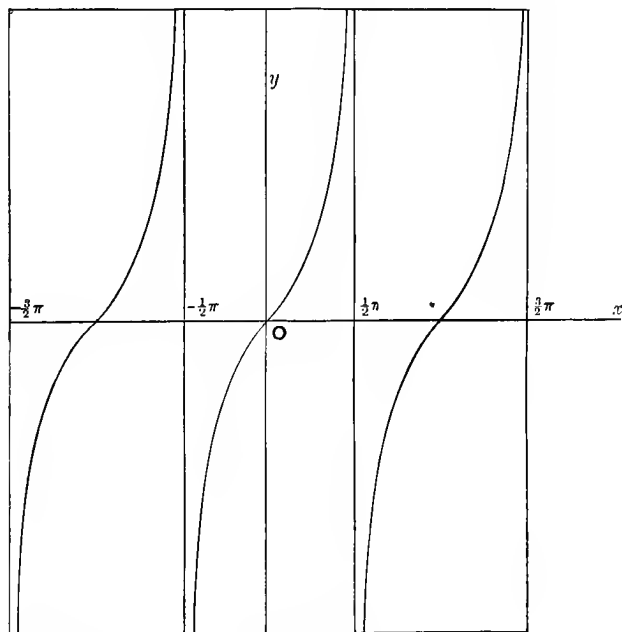


$y = \tan x$.

Fig. 47.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$\tan x$	0	0.100	0.203	0.309	0.423	0.546	0.684	0.842

x	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$\tan x$	1.030	1.260	1.557	1.965	2.572	3.602	5.798	14.101



$$y = \tan x.$$

Fig. 48.

EXAMPLES.

1. How do the graphs of $\sin mx$ and $\cos mx$ differ from those of $\sin x$ and $\cos x$ (i) when $m > 1$, (ii) when $m < 1$?

2. How do the graphs of $a \sin x$ and $a \cos x$ differ from those of $\sin x$ and $\cos x$ (i) when $a > 1$, (ii) when $a < 1$?

3. Draw roughly the graphs of $\frac{1}{2} \cos 2x$ and $2 \sin \frac{1}{2}x$.

4. Draw the graph of $\sin x - \cos x$.

5. For what values of x is $\tan x$ equal to 1? for what values is it equal to -1 ?

109. With a view to the differentiation of $\sin \theta$ and $\cos \theta$ we shall think of the sine and cosine of an angle of θ radians, and, in the first instance, we shall take θ to lie between 0 and $\frac{1}{2}\pi$.

The method which will be explained here turns upon two observations.

(1) If P, Q are two points very near together on a circle (Fig. 49), χ units of length the distance PQ, and l units of length the length of the arc PQ, then χ is very nearly equal to l , and, when Q approaches P, so that χ tends to zero, $\frac{l}{\chi}$ tends to 1 as a limit.

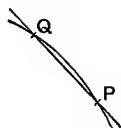


Fig. 49.

This result is really involved in the radian measure of angles. A formal proof will be found in Appendix VI.

(2) In Fig. 50 let P, Q be two points on a circle, O the centre of the circle. Draw the axes of x and y through O, draw the ordinates PN, QM, and the straight line PR parallel to the axis of x to meet QM in R. Produce this line to the right, as shown, draw the tangent PT, as shown, and let the axis of x cut the circle in A, as shown. Let the angles $\angle xOP$, $\angle x'PQ$, $\angle x'PT$ be θ , β , ϕ radians. Let the lengths of the arcs AP and AQ be s and $s + \Delta s$ units of length, let the coordinates of P be x , y and those

of Q be $x + \Delta x$, $y + \Delta y$, and let the length of the chord PQ be χ units of length. Then

$$\frac{\Delta x}{\chi} = \cos \beta, \quad \frac{\Delta y}{\chi} = \sin \beta,$$

for Δx and Δy have the right signs as well as the right magnitudes.

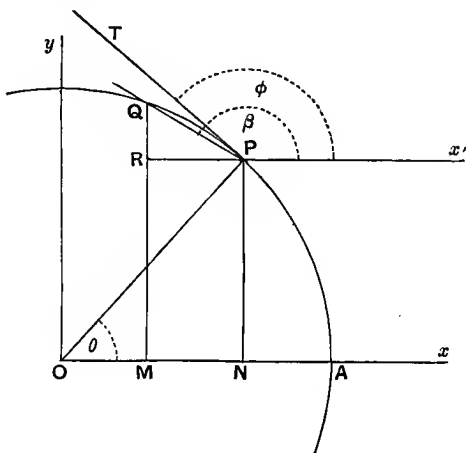


Fig. 50.

110. We can now differentiate $\sin \theta$ and $\cos \theta$. We have

$$x = r \cos \theta, \quad y = r \sin \theta, \quad s = r\theta,$$

where r is independent of θ , and therefore

$$\frac{dx}{ds} = \frac{d(\cos \theta)}{d\theta}, \quad \frac{dy}{ds} = \frac{d(\sin \theta)}{d\theta}.$$

Now $\frac{dx}{ds}$ and $\frac{dy}{ds}$ are the limits to which $\frac{\Delta x}{\Delta s}$ and $\frac{\Delta y}{\Delta s}$ tend when

Δs tends to zero. But

$$\frac{\Delta x}{\Delta s} = \frac{\Delta x}{\chi} \cdot \frac{\chi}{\Delta s} = \cos \beta \frac{\chi}{\Delta s}, \quad \frac{\Delta y}{\Delta s} = \frac{\Delta y}{\chi} \cdot \frac{\chi}{\Delta s} = \sin \beta \frac{\chi}{\Delta s},$$

and the factors on the right all tend to known limits, for the limit of β is ϕ , and the limit of $\frac{\chi}{\Delta s}$ is 1. Hence

$$\frac{dx}{ds} = \cos \phi, \quad \frac{dy}{ds} = \sin \phi.$$

Further $\phi = \frac{1}{2}\pi + \theta$, so that $\cos \phi = -\sin \theta$, $\sin \phi = \cos \theta$.
Therefore

$$\frac{d(\cos \theta)}{d\theta} = -\sin \theta, \quad \frac{d(\sin \theta)}{d\theta} = \cos \theta \quad \dots\dots(a).$$

111. We have proved the two formulae (a) by assuming that Δs is positive. When Δs is negative, as in Fig. 51, we have

$$\frac{\Delta x}{\chi} = -\cos \beta', \quad \frac{\Delta y}{\chi} = -\sin \beta',$$

where the angle $x'PQ$ is $\pi - \beta'$ radians; and the limit to which $\frac{\Delta s}{\chi}$ tends is -1 .

As before $\frac{dx}{ds} = \cos \phi$, and $\frac{dy}{ds} = \sin \phi$, so that the formulae (a) are unaltered.

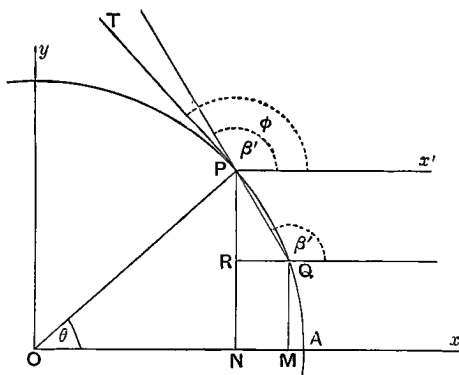


Fig. 51.

112. We may extend the formulae

$$\frac{d(\sin \theta)}{d\theta} = \cos \theta, \quad \frac{d(\cos \theta)}{d\theta} = -\sin \theta$$

to other values of θ by using the properties of the functions. When θ is between $\frac{1}{2}\pi$ and π , we may put $\theta = \frac{1}{2}\pi + \theta'$; then θ' is between 0 and $\frac{1}{2}\pi$, and we use the formulae (1) of § 102. We find

$$\begin{aligned}\frac{d(\sin \theta)}{d\theta} &= \frac{d(\cos \theta')}{d\theta'} \frac{d\theta'}{d\theta} = -\sin \theta' = \cos \theta, \\ \frac{d(\cos \theta)}{d\theta} &= -\frac{d(\sin \theta')}{d\theta'} \frac{d\theta'}{d\theta} = -\cos \theta' = -\sin \theta.\end{aligned}$$

The formulae are now proved for all values of θ between 0 and π . When θ is between π and 2π , we may put $\theta = \pi + \theta'$; then θ' is between 0 and π , and we use the formulae (2) of § 102. We find

$$\begin{aligned}\frac{d(\sin \theta)}{d\theta} &= -\frac{d(\sin \theta')}{d\theta'} \frac{d\theta'}{d\theta} = -\cos \theta' = \cos \theta, \\ \frac{d(\cos \theta)}{d\theta} &= -\frac{d(\cos \theta')}{d\theta'} \frac{d\theta'}{d\theta} = \sin \theta' = -\sin \theta.\end{aligned}$$

The formulae are now proved for all values of θ between 0 and 2π .

113. In the further extension of the results we shall not regard the variable as the numerical measure of an angle measured in radians, but simply as a number. We know now that, if x has any value between 0 and 2π ,

$$\frac{d(\sin x)}{dx} = \cos x, \quad \frac{d(\cos x)}{dx} = -\sin x \quad \dots\dots\dots (a).$$

We can at once extend the result to all positive values of x . If x is greater than 2π we can put $x = 2n\pi + x'$, where x' is between 0 and 2π , and n is some integer. Then we have, for example,

$$\frac{d(\sin x)}{dx} = \frac{d(\sin x')}{dx} = \frac{d(\sin x')}{dx'} \frac{dx'}{dx} = \frac{d(\sin x')}{dx'} = \cos x' = \cos x,$$

and similarly for the cosine.

To extend the result to negative values of x we put $x = -x'$. Then x' is positive. We have, for example,

$$\begin{aligned}\frac{d(\sin x)}{dx} &= \frac{d\{\sin(-x')\}}{dx} = -\frac{d(\sin x')}{dx} = -\frac{d(\sin x')}{dx'} \frac{dx'}{dx} = -\frac{d(\sin x')}{dx'} \\ &= \cos x' = \cos x,\end{aligned}$$

and similarly for the cosine.

The formulae (a) are now established¹ for all values of x .

¹ Another method of proof will be found in Appendix VI.

114. By using the defining equations (1) of § 108 and the rules of differentiation we may prove the results

$$\frac{d(\tan x)}{dx} = \sec^2 x, \quad \frac{d(\sec x)}{dx} = \sec x \tan x,$$

$$\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x, \quad \frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x.$$

Take, for instance, $\frac{d(\tan x)}{dx}$, and use the rule for differentiating a quotient (§ 28). We have

$$\tan x = \frac{\sin x}{\cos x},$$

and therefore

$$\begin{aligned} \frac{d(\tan x)}{dx} &= \frac{\cos x \frac{d(\sin x)}{dx} - \sin x \frac{d(\cos x)}{dx}}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}. \end{aligned}$$

115. To all these formulae of differentiation there correspond formulae of integration. The most important are

$$\int \sin x \, dx = -\cos x \quad (\text{D}),$$

$$\int \cos x \, dx = \sin x \quad (\text{E}),$$

$$\int \sec^2 x \, dx = \tan x \quad (\text{F}).$$

These are standard forms.

EXAMPLES

In these examples a, b, n, α, β are constants.

1. Differentiate the following (1)–(10) :—

- (1) $a \sin nx$, (2) $a \cos nx$, (3) $x \sin x$, (4) $x^2 \cos x$,
 (5) $\frac{\sin x}{x}$, (6) $\tan^2 x$, (7) $x + \sin x \cos x$, (8) $x - \sin x \cos x$,
 (9) $\log(\sin x)$, (10) $\log(\cos x)$.

2. Prove that, if h is very small,

- (1) $\sin(x+h) \doteq \sin x + h \cos x$, (2) $\cos(x+h) \doteq \cos x - h \sin x$,
 (3) $\tan(x+h) \doteq \tan x + h \sec^2 x$.

Find in each case what the result becomes if $x=0$.

3. A straight line subtends an angle of 2α radians at the centre of a circle of radius r units of length. Prove that its length is $2r \sin \alpha$ units of length. Hence, assuming the proposition assumed in § 109 (1), prove that, as α tends to zero, $\frac{\sin \alpha}{\alpha}$ tends to 1 as a limit. Deduce that, when α is very small, $\sin \alpha$ is nearly equal to α . [Observe that this result is in accord with the table of § 105, which shows how nearly $\sin \alpha$ approaches α when $\alpha < 0.3$.]

4. The top of a perpendicular cliff is observed from a point on a level with the foot of the cliff at a distance of 100 yards from the foot, and the angle of elevation is estimated as 40° . If the angle is estimated correctly, find the height of the cliff. If the angle can be estimated correctly to half a degree, find approximately the error of the calculated height. [Result 4.5 feet.]

5. Prove that the maximum value of $a \cos nx + b \sin nx$ is $\sqrt{a^2 + b^2}$, and the minimum value is $-\sqrt{a^2 + b^2}$.

6. Integrate the following (1)–(7) :—

- (1) $\sin nx$, (2) $\cos nx$, (3) $\cos^2 x$, (4) $\sin^2 x$,
 (5) $\tan^2 x$, (6) $\tan x$, (7) $\cot x$.

7. Prove that, if $x = a \cos(nt - \beta)$, $\frac{d^2x}{dt^2} + n^2x = 0$.

8. Prove that $\frac{d(e^{ax} \cos nx)}{dx} = e^{ax} (a \cos nx - n \sin nx)$,

and that $\frac{d(e^{ax} \sin nx)}{dx} = e^{ax} (a \sin nx + n \cos nx)$.

Deduce that
$$\int e^{ax} \cos nx \, dx = e^{ax} \frac{a \cos nx + n \sin nx}{a^2 + n^2}.$$

and that
$$\int e^{ax} \sin nx \, dx = e^{ax} \frac{a \sin nx - n \cos nx}{a^2 + n^2}.$$

9. Find the area contained between the axis of x and one bay of the curve of sines whose equation is $y = a \sin nx$.

10. Prove that the function $e^{-x} \cos \left(x - \frac{1}{4} \pi \right)$ has a maximum value when $x=0$ and also whenever x is a multiple of 2π , and that it has a minimum value whenever x is an odd multiple of π .

APPLICATIONS TO OSCILLATORY MOTIONS

116. Many of the most important applications of trigonometric functions arise in connexion with oscillatory motions. In the simplest of oscillatory motions a point of a body moves to and fro in a straight line (which we may take to be the axis of x), according to the law expressed by the equation $x = a \cos nt$, where a and n are constants. At first the x -coordinate of the point is a , which we take to be positive. The motion is such that x begins to diminish, and at the instant specified by $t = \frac{\pi}{2n}$, x is equal to zero; x then becomes negative, and the point moves through the origin in the negative sense of the axis of x , and comes to instantaneous rest, at the point specified by $x = -a$, at the instant specified by $t = \frac{\pi}{n}$. It then moves back again, passing through the origin at the instant specified by $t = \frac{3\pi}{2n}$, and comes again to instantaneous rest, at the point specified by $x = a$, at the instant specified by $t = \frac{2\pi}{n}$. The whole motion is then repeated. If the unit of time is a second, the motion from one extreme position to the other and back again takes $\frac{2\pi}{n}$ seconds.

A motion of this kind is called a "simple harmonic motion," and the time $\frac{2\pi}{n}$ seconds a "complete period."

If the formula were $x = a \cos (nt - \beta)$ instead of $x = a \cos nt$, the motion would still be simple harmonic motion, the period would be $\frac{2\pi}{n}$ seconds, and the extreme positions would be given by $x = a$ and $x = -a$, but they would be attained at the instants given by $t = \frac{\beta}{n}$ and $t = \frac{\beta + N\pi}{n}$, where N is any positive integer. The position $x = a$ would be attained when the N in this formula is even, the position $x = -a$ when the N in this formula is odd.

117. We consider the problem of determining the motion of a body attached to a spring. We shall suppose the line of the spring to be horizontal, and the body to be supported so that it can move in the line of the spring, and we shall at first neglect all resistances. Let the line of motion be the axis of x , and the origin the position occupied by the centre of gravity of the body when the spring has its natural length. If the body in this position has no velocity it remains at rest. Let the positive sense of the axis of x be that of extension of the spring. We draw it towards the right.

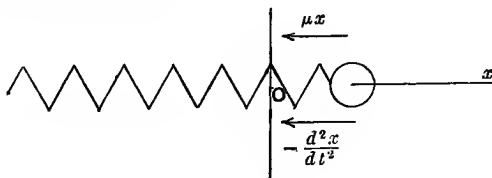


Fig. 52.

When the spring is extended, so that the body is displaced x feet to the right, the spring exerts a pull towards the left. The pull is proportional to x , and we may take it to be μx lbs., where μ is a constant depending upon the spring. This force produces

an acceleration towards the left. We have seen (§ 81) that when there is acceleration towards the left its amount is $-\frac{d^2x}{dt^2}$ units (Fig. 52). Now taking the weight of the body to be W lbs.¹, we know that, if the body were free, the force of W lbs. acting on it would produce in it an acceleration of 32.2 foot-second units, or, as we write it, g units. Also we know that the acceleration produced in a body by a force is proportional to the force producing it. We therefore have, in the above case

$$-\frac{d^2x}{dt^2} : g = \mu x : W,$$

or

$$\frac{d^2x}{dt^2} = -\frac{g\mu x}{W}.$$

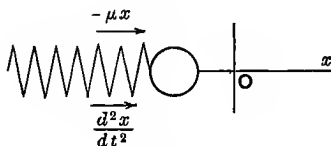


Fig. 53.

Again, when the spring is contracted, x is negative, and the spring exerts on the body a push of $\mu(-x)$ lbs. This push produces an acceleration towards the right, and its amount is $\frac{d^2x}{dt^2}$ units (Fig. 53). By the same reasoning as before we have

$$\frac{d^2x}{dt^2} : g = -\mu x : W,$$

which gives the same equation as before. If we put

$$n^2 = \frac{g\mu}{W},$$

the equation becomes

$$\frac{d^2x}{dt^2} + n^2x = 0,$$

and this equation holds for all the values of x that can occur.

¹ In regard to the units see Appendix VII.

Now we know that this equation can be satisfied by putting

$$x = a \cos (nt - \beta) \quad (1).$$

See Ex. 7, p. 114. This form is, as we shall see presently, sufficiently general to admit of the body having at the instant from which t is reckoned any given displacement and velocity. Let these be taken so that

$$x = a \text{ and } \frac{dx}{dt} = u \text{ when } t = 0.$$

Now if $x = a \cos (nt - \beta)$ in general,

$$x = a \cos \beta \text{ and } \frac{dx}{dt} = na \sin \beta \text{ when } t = 0,$$

and therefore we have to find a and β so that

$$a \cos \beta = a, \quad a \sin \beta = \frac{u}{n}.$$

This can always be done (see § 104). We conclude that the formula (1) may represent the value of x at any time. It can be proved (see Ex. 5, p. 122) that no other formula can satisfy all the conditions. The body therefore executes a simple harmonic motion of period $\frac{2\pi}{n}$.

118. Again consider the case where the body supported by the spring is subject to a resistance proportional to the velocity. Let the resisting force be λv lbs. when the velocity is v feet per second. When $\frac{dx}{dt}$ is positive, or the body is moving to the right, this force produces an acceleration towards the left, of amount $\frac{\lambda g}{W} \frac{dx}{dt}$ units. This is to be added to the acceleration produced by the pull or push of the spring. We have therefore two contributions to $\frac{d^2x}{dt^2}$, one is $-\frac{\lambda g}{W} \frac{dx}{dt}$ and the other is $-\frac{\mu g}{W} x$, and thus we get the equation

$$\frac{d^2x}{dt^2} = -\frac{\lambda g}{W} \frac{dx}{dt} - \frac{\mu g}{W} x.$$

If the body is moving to the left, so that $\frac{dx}{dt}$ is negative, the resistance produces an acceleration towards the right, of amount $\frac{\lambda g}{W} \left(-\frac{dx}{dt} \right)$ units, and we get the same equation as before. Writing

$$\frac{\lambda g}{W} = k, \quad \frac{\mu g}{W} = n^2,$$

we have the equation in the form

$$\frac{d^2x}{dt^2} + k \frac{dx}{dt} + n^2x = 0.$$

Now we know (Ex. 9, p. 92) that this equation can be written

$$\frac{d^2(xe^{\frac{1}{2}kt})}{dt^2} + \left(n^2 - \frac{1}{4} k^2 \right) xe^{\frac{1}{2}kt} = 0.$$

The solution of this equation takes different forms according as $n^2 - \frac{1}{4} k^2 > 0$ or < 0 .

When the resistance is not very great, so that $\frac{1}{4} k^2 < n^2$, we put

$$n^2 - \frac{1}{4} k^2 = m^2.$$

Then
$$\frac{d^2(xe^{\frac{1}{2}kt})}{dt^2} + m^2(xe^{\frac{1}{2}kt}) = 0,$$

and we know that $xe^{\frac{1}{2}kt} = a \cos(mt - \beta),$

where a and β are constants depending on the initial displacement and initial velocity. Hence x is given in terms of t by an equation of the form

$$x = ae^{-\frac{1}{2}kt} \cos(mt - \beta).$$

The motion expressed by this equation is oscillatory. The body moves alternately from right to left and from left to right in a period of $\frac{2\pi}{m}$ seconds, but the extreme values of the displacement

continually diminish owing to the presence of the exponential factor $e^{-\frac{1}{2}kt}$. The body settles down, as it were, into its position of equilibrium, by oscillating about it with excursions of continually diminishing amount.

Most natural oscillations take place in such a way as this. A motion of this kind would be described as a "damped harmonic motion."

119. When the resistance is very great the motion is quite different. If $\frac{1}{4}k^2 > n^2$, we put

$$\frac{1}{4}k^2 - n^2 = m^2.$$

Then
$$\frac{d^2(xe^{\frac{1}{2}kt})}{dt^2} - m^2(xe^{\frac{1}{2}kt}) = 0.$$

Now we know (Ex. 8, p. 92) that this equation can be satisfied by putting

$$xe^{\frac{1}{2}kt} = Ae^{mt} + Be^{-mt},$$

where A and B are constants, and it can be proved (see § 127 below) that the equation cannot be satisfied in any other way. Hence we have

$$x = e^{-\frac{1}{2}kt}(Ae^{mt} + Be^{-mt}).$$

We consider the case where the body is at first displaced a feet to the right, held for a moment, and then let go. Then we have

$$x = a \text{ and } \frac{dx}{dt} = 0 \text{ when } t = 0.$$

Now if
$$x = e^{-\frac{1}{2}kt}(Ae^{mt} + Be^{-mt}),$$

we find
$$a = A + B, \quad 0 = A\left(-\frac{1}{2}k + m\right) + B\left(-\frac{1}{2}k - m\right),$$

so that
$$A = \frac{1}{2}a\left(\frac{k}{2m} + 1\right), \quad B = -\frac{1}{2}a\left(\frac{k}{2m} - 1\right),$$

and
$$x = \frac{1}{2}ae^{-\frac{1}{2}kt}\left\{\left(\frac{k}{2m} + 1\right)e^{mt} - \left(\frac{k}{2m} - 1\right)e^{-mt}\right\}.$$

Since $n^2 < \frac{1}{4}k^2$ and $m^2 = \frac{1}{4}k^2 - n^2$, $\frac{1}{4}k^2 - m^2 = n^2$ and $\frac{1}{2}k > m$, or $\frac{k}{2m} > 1$. We see that x is always positive, a being positive. Again we find

$$\frac{dx}{dt} = -\frac{1}{2}am\left(\frac{k}{2m} + 1\right)\left(\frac{k}{2m} - 1\right)e^{-\frac{1}{2}kt}(e^{mt} - e^{-mt}),$$

and this formula shows that $\frac{dx}{dt}$ is always negative. The body therefore always moves towards the position of equilibrium, but with continually diminishing velocity. It creeps towards this position, but never quite reaches it.

120. A special case of the problem concerning electric currents, considered in § 95, arises when the impressed electromotive force is periodic. We shall suppose that, in the notation of § 95, $E = A \cos pt$, where A and p are constants, and that $i = 0$ when $t = 0$. Then we have the equation $L \frac{di}{dt} + Ri = E$, or

$$\frac{di}{dt} + bi = \frac{A}{L} \cos pt.$$

This is the same as

$$\frac{d(ie^{bt})}{dt} = \frac{A}{L} e^{bt} \cos pt,$$

and (Ex. 8, p. 114) this gives

$$ie^{bt} = \frac{A}{L} e^{bt} \frac{p \sin pt + b \cos pt}{p^2 + b^2} + C.$$

To make $i = 0$ when $t = 0$ we must have

$$\frac{Ab}{L(p^2 + b^2)} + C = 0,$$

and therefore

$$ie^{bt} = \frac{A}{L} \left(e^{bt} \frac{p \sin pt + b \cos pt}{p^2 + b^2} - \frac{b}{p^2 + b^2} \right),$$

$$\text{or} \quad i = \frac{A}{L(p^2 + b^2)} (p \sin pt + b \cos pt - be^{-bt}).$$

On substituting $\frac{R}{L}$ for b , we find

$$i = \frac{A}{R^2 + p^2 L^2} \left(pL \sin pt + R \cos pt - Re^{-\frac{Rt}{L}} \right).$$

This formula gives the intensity of the alternating current produced in a circuit by a periodic electromotive force, the circuit being closed at the instant specified by $t = 0$, when the electromotive force is a maximum.

EXAMPLES

1. A point P moves on a circle (centre C) in such a way that the straight line CP turns with a uniform angular velocity, and from P a perpendicular PN is let fall upon a fixed diameter. Prove that N executes a simple harmonic motion.

2. Draw the displacement-time graph and the velocity-time graph for a point which executes a simple harmonic motion.

3. A body executes a simple harmonic motion according to the equation

$$\frac{d^2x}{dt^2} + n^2x = 0.$$

Initially it is displaced so that $x=a$, and it is let go from rest in this position. Express x in terms of t .

Initially the body is undisplaced and it is projected so that $\frac{dx}{dt}=u$.

Express x in terms of t .

4. If $\frac{d^2x}{dt^2} + n^2x = 0$, we may put $\frac{dx}{dt} = v$, and then $\frac{dv}{dt} = -n^2x$. Prove that $v \frac{dv}{dt} + n^2x \frac{dx}{dt} = 0$, and thence that $\left(\frac{dx}{dt}\right)^2 + n^2x^2 = \text{const.}$

Prove also that, if a body moves according to the equation

$$\left(\frac{dx}{dt}\right)^2 + n^2x^2 = \text{const.}$$

it executes a simple harmonic motion.

5. Prove that if two functions satisfy the equation $\frac{d^2x}{dt^2} + n^2x = 0$, and yield the same values for x and $\frac{dx}{dt}$ when $t=0$, their difference z satisfies the equation $\frac{d^2z}{dt^2} + n^2z = 0$ and yields zero values for z and $\frac{dz}{dt}$ when $t=0$.

Proceed to prove that $\left(\frac{dz}{dt}\right)^2 + n^2z^2 = 0$, and thence that $z=0$ for all positive values of t . Hence show that there is only one function which satisfies the conditions laid down for x .

6. A body executes a damped harmonic motion according to the equa-

tion $x = ae^{-\frac{1}{2}kt} \cos(mt - \beta)$. Prove that if $x = a$ and $\frac{dx}{dt} = 0$ when $t = 0$, $a = a\sqrt{1 + \frac{k^2}{4m^2}}$, and $\tan \beta = \frac{k}{2m}$. Prove that the maxima of the displacement (sign disregarded) occur when $t = 0$ or $t = a$ multiple of $\frac{\pi}{m}$, and that each maximum is less than the preceding in the ratio $e^{-\frac{\pi k}{2m}} : 1$ (sign disregarded). Prove further that the body passes through its equilibrium position at a series of instants separated by intervals of $\frac{\pi}{m}$ seconds (a second being the unit of time), and determine the first of these instants. Prove that, if any one of these instants is specified by the equation $t = T$, the velocity at that instant is $mae^{-\frac{1}{2}kT}$ feet per second (a foot being the unit of length).

7. Draw the displacement-time graph for the body whose motion is described in Ex. 6. [Draw the locus of the maxima $x = ae^{-\frac{1}{2}kt}$ and $x = -ae^{-\frac{1}{2}kt}$, the graph is like a curve of sines with steadily diminishing maximum ordinates whose extremities lie alternately on these two curves.]

8. A gate supported on hinges swings to and fro, starting from rest in an extreme position in which its angular displacement from its equilibrium position is 45° , and next coming to rest in an extreme position in which its angular displacement is 40° on the other side of the equilibrium position, the time between these two positions of rest being 1 second. Assuming that its angular displacement θ obeys the law of damped harmonic motion [$\theta = ae^{-\frac{1}{2}kt} \cos(mt - \beta)$], determine the constants a , β , $\frac{1}{2}k$, m . [Results, $\frac{1}{2}k \doteq 0.1178$, $m = \pi$, $\beta \doteq 0.0375$, $a \doteq 0.7909$.]

Find the angular velocity with which the gate first passes through its position of equilibrium, and the ratio in which this angular velocity is reduced at each subsequent passage. [Results: angular velocity = 2.34 radians per second approximately. Reduction $\frac{8}{9}$.]

9. A point, which executes a damped harmonic motion, is observed to come to instantaneous rest in three successive positions, which are respectively 10 inches to the right of a point A, 9 inches to the left, 8 inches to the right. Find its equilibrium position. [Result $\frac{1}{36}$ of an inch to the left of A.]

10. In the problem of § 120 after a time the term $Re^{-\frac{Rt}{L}}$ in the expression for i becomes very small and may be omitted. Prove that, when this is done, the maximum value of i is $\frac{A}{\sqrt{(R^2 + p^2 L^2)}}$, and that the maxima of current follow the maxima of electromotive force after intervals of s seconds, where s is the smallest positive number which satisfies the equation $\tan ps = \frac{pL}{R}$.

CHAPTER VIII

METHODS OF INTEGRATION

It is very important to be able to integrate such simple functions as can be integrated easily. A number of methods which are applicable to various classes of functions will be exemplified.

THE METHOD OF RESOLUTION INTO PARTIAL FRACTIONS.

121. We begin with three examples,

$$(1) \quad \int \frac{1}{1-x^2} dx.$$

We know that $1-x^2=(1-x)(1+x)$, and we know how to integrate the sum $\frac{1}{1+x} + \frac{1}{1-x}$. Now

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2},$$

therefore
$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \int \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx$$
$$= \frac{1}{2} \{ \log_e (1+x) - \log_e (1-x) \} = \frac{1}{2} \log_e \frac{1+x}{1-x}.$$

$$(2) \quad \int \frac{1}{(x-1)(x-2)} dx.$$

Here we observe that the difference

$$\frac{1}{x-2} - \frac{1}{x-1} = \frac{1}{(x-1)(x-2)},$$

and therefore the required integral is $\log_e \frac{x-2}{x-1}$.

(3) We observe that $\frac{1}{x-1} + \frac{1}{x-2} = \frac{2x-3}{(x-1)(x-2)},$

and therefore $\int \frac{2x-3}{(x-1)(x-2)} dx = \log_e \{(x-1)(x-2)\}.$

From this result and that in Ex. 2 we see that

$$\begin{aligned} \int \frac{2x}{(x-1)(x-2)} dx &= \log_e \{(x-1)(x-2)\} + 3 \log_e \frac{x-2}{x-1} \\ &= 4 \log_e (x-2) - 2 \log_e (x-1), \end{aligned}$$

so that $\int \frac{x}{(x-1)(x-2)} dx = 2 \log_e (x-2) - \log_e (x-1).$

122. From the above examples we can see how to integrate any expression of the form $\frac{Ax+B}{(x-a)(x-b)}.$ We form the sum $\frac{1}{x-a} + \frac{1}{x-b}$ and the difference $\frac{1}{x-a} - \frac{1}{x-b}.$ We get

$$\frac{1}{x-a} + \frac{1}{x-b} = \frac{2x-(a+b)}{(x-a)(x-b)}, \quad \frac{1}{x-a} - \frac{1}{x-b} = \frac{a-b}{(x-a)(x-b)},$$

and we see that

$$Ax+B = \frac{1}{2} A \{2x-(a+b)\} + \left\{ B + \frac{1}{2} A(a+b) \right\},$$

and therefore

$$\begin{aligned} \frac{Ax+B}{(x-a)(x-b)} &= \frac{A}{2} \left(\frac{1}{x-a} + \frac{1}{x-b} \right) + \frac{B + \frac{1}{2} A(a+b)}{a-b} \left(\frac{1}{x-a} - \frac{1}{x-b} \right) \\ &= \frac{1}{x-a} \left\{ \frac{A}{2} \left(1 + \frac{a+b}{a-b} \right) + \frac{B}{a-b} \right\} \\ &\quad + \frac{1}{x-b} \left\{ \frac{A}{2} \left(1 - \frac{a+b}{a-b} \right) - \frac{B}{a-b} \right\} \\ &= \frac{Aa+B}{a-b} \frac{1}{x-a} - \frac{Ab+B}{a-b} \frac{1}{x-b}. \end{aligned}$$

Hence

$$\frac{Ax+B}{(x-a)(x-b)} = \frac{Aa+B}{a-b} \frac{1}{x-a} + \frac{Ab+B}{b-a} \frac{1}{x-b}.$$

In this formula the expression in the left-hand member is "resolved into partial fractions"; it is expressed as the sum of a constant multiplied by $\frac{1}{x-a}$ and another constant multiplied by $\frac{1}{x-b}$. We have now

$$\int \frac{Ax + B}{(x-a)(x-b)} dx = \frac{Aa + B}{a-b} \log_e (x-a) + \frac{Ab + B}{b-a} \log_e (x-b).$$

123. By way of extension of this method we consider the integral

$$\int \frac{1}{(x-a)(x-b)(x-c)} dx.$$

If we subtract from $\frac{1}{(x-a)(x-b)(x-c)}$ an expression of the form $\frac{C}{x-c}$ we find

$$\frac{1}{(x-a)(x-b)(x-c)} - \frac{C}{x-c} = \frac{1-C(x-a)(x-b)}{(x-a)(x-b)(x-c)},$$

and we can adjust C so that the numerator of the right-hand member may be divisible by $x-c$. We have only to make

$$C(c-a)(c-b) = 1, \text{ or } C = \frac{1}{(c-a)(c-b)}.$$

Thus

$$\begin{aligned} \frac{1}{(x-a)(x-b)(x-c)} &= \frac{1}{(c-a)(c-b)(x-c)} \\ &\quad - \frac{1}{(c-a)(c-b)} \frac{(x-a)(x-b) - (c-a)(c-b)}{(x-a)(x-b)(x-c)}. \end{aligned}$$

Now

$$(x-a)(x-b) - (c-a)(c-b) = x^2 - c^2 - (a+b)(x-c),$$

and therefore

$$\begin{aligned} \frac{1}{(x-a)(x-b)(x-c)} &= \frac{1}{(c-a)(c-b)(x-c)} \\ &\quad - \frac{x+c-(a+b)}{(c-a)(c-b)(x-a)(x-b)}. \end{aligned}$$

Further by the result obtained in § 122

$$\begin{aligned}\frac{x+c-(a+b)}{(x-a)(x-b)} &= \frac{a+c-(a+b)}{a-b} \frac{1}{x-a} + \frac{b+c-(a+b)}{b-a} \frac{1}{(x-b)} \\ &= \frac{c-b}{a-b} \frac{1}{x-a} + \frac{c-a}{b-a} \frac{1}{x-b}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{(x-a)(x-b)(x-c)} &= \frac{1}{(c-a)(c-b)(x-c)} \\ &\quad + \frac{1}{(a-c)(a-b)(x-a)} + \frac{1}{(b-a)(b-c)(x-b)}.\end{aligned}$$

The expression to be integrated is resolved into a sum of three partial fractions, each of which we know how to integrate.

The method which has been explained here admits of extension to any rational function whose denominator is given as a product of factors, or can be resolved into a product of factors.

124. As an additional example we consider the integral

$$\frac{ax+\beta}{x^2+px+q} dx.$$

If we could resolve the denominator into factors of the form $x+a$ and $x+b$, we could integrate the given expression by the method of resolution into partial fractions. We proceed, just as if we were going to solve the quadratic equation $x^2+px+q=0$, by adding $\frac{1}{4}p^2$ and then subtracting it.

$$\text{We have} \quad x^2+px+q = x^2+px+\frac{1}{4}p^2 - \left(\frac{1}{4}p^2 - q\right).$$

$$\text{If} \quad p^2 > 4q,$$

we can put

$$\sqrt{\left(\frac{1}{4}p^2 - q\right)} = \gamma,$$

and then

$$\begin{aligned}x^2+px+q &= \left(x + \frac{1}{2}p\right)^2 - \gamma^2 \\ &= \left\{\left(x + \frac{1}{2}p\right) + \gamma\right\} \left\{\left(x + \frac{1}{2}p\right) - \gamma\right\},\end{aligned}$$

and if

$$a = \frac{1}{2}p + \gamma, \quad b = \frac{1}{2}p - \gamma,$$

we have

$$x^2+px+q = (x+a)(x+b).$$

We may proceed by a method which is slightly different from that used in § 122, by putting $x + \frac{1}{2}p = z$.

Then we have
$$\frac{ax + \beta}{x^2 + px + q} = \frac{az + \beta - \frac{1}{2}ap}{z^2 - \gamma^2},$$

and
$$\int \frac{ax + \beta}{x^2 + px + q} dx = a \int \frac{z}{z^2 - \gamma^2} dz + \left(\beta - \frac{1}{2}ap \right) \int \frac{1}{z^2 - \gamma^2} dz,$$

but
$$\int \frac{z}{z^2 - \gamma^2} dz = \frac{1}{2} \log_e (z^2 - \gamma^2) = \frac{1}{2} \log_e (x^2 + px + q),$$

and
$$\int \frac{1}{z^2 - \gamma^2} dz = \frac{1}{2\gamma} \log_e \frac{z - \gamma}{z + \gamma},$$

also
$$2\gamma = a - b, \quad z - \gamma = x + b, \quad z + \gamma = x + a,$$

hence
$$\int \frac{ax + \beta}{x^2 + px + q} dx = \frac{1}{2} a \log_e (x^2 + px + q) + \frac{\beta - \frac{1}{2}ap}{a - b} \log_e \frac{x + b}{x + a}.$$

The case where $p^2 < 4q$ will be considered in § 133.

EXAMPLES

Integrate the following (1)–(10):—

- (1) $\frac{1}{x^2 - 4}$, (2) $\frac{1}{x^2 - 5}$, (3) $\frac{1}{4x^2 - 1}$, (4) $\frac{1}{5x^2 - 1}$,
 (5) $\frac{2x + 3}{(x + 1)(x - 2)}$, (6) $\frac{x - 3}{(1 - x)(2x - 1)}$, (7) $\frac{1}{x(x - 1)}$, (8) $\frac{1}{x(x - 1)(x - 2)}$,
 (9) $\frac{3x + 4}{x^2 + 3x + 2}$, (10) $\frac{3x + 4}{x^2 + 3x + 1}$.

A LOGARITHMIC FORMULA

125. The result found in Ex. 2, p. 92 gives us an important integral, viz.:

$$\int \frac{1}{\sqrt{(x^2 + C)}} dx = \log_e \{x + \sqrt{(x^2 + C)}\} \quad (G).$$

This is a standard form. It is valid whether C is positive or negative.

126. We can deduce the value of the integral

$$\int \frac{1}{\sqrt{(x^2 + px + q)}} dx.$$

We may put

$$x^2 + px + q = \left(x + \frac{1}{2}p\right)^2 + \left(q - \frac{1}{4}p^2\right),$$

and

$$x + \frac{1}{2}p = z,$$

then

$$\begin{aligned} \int \frac{1}{\sqrt{(x^2 + px + q)}} dx &= \int \frac{1}{\sqrt{z^2 + \left(q - \frac{1}{4}p^2\right)}} dz \\ &= \log_e \left[z + \sqrt{\left\{z^2 + \left(q - \frac{1}{4}p^2\right)\right\}} \right], \end{aligned}$$

but

$$z^2 + q - \frac{1}{4}p^2 = x^2 + px + q,$$

therefore

$$\int \frac{1}{\sqrt{(x^2 + px + q)}} dx = \log_e \left\{ x + \frac{1}{2}p + \sqrt{(x^2 + px + q)} \right\}.$$

127. By means of the standard form (G) we may express y in terms of t when y satisfies the equation $\frac{d^2y}{dt^2} = m^2y$, where m is constant.

Put

$$w = \frac{dy}{dt},$$

so that

$$\frac{dw}{dt} = m^2y,$$

and

$$w \frac{dw}{dt} = m^2y \frac{dy}{dt},$$

or

$$\frac{d\left(\frac{1}{2}w^2 - \frac{1}{2}m^2y^2\right)}{dt} = 0.$$

Then $w^2 - m^2y^2$ is a constant and we may put

$$w^2 - m^2y^2 = m^2\mathbf{C},$$

where \mathbf{C} is a constant. Thus we have

$$\left(\frac{dy}{dt}\right)^2 = m^2(y^2 + \mathbf{C}).$$

Hence either $m \frac{dt}{dy} = \frac{1}{\sqrt{(y^2 + C)}} \quad \text{or} \quad m \frac{dt}{dy} = -\frac{1}{\sqrt{(y^2 + C)}}.$

First suppose that $m \frac{dt}{dy} = \frac{1}{\sqrt{(y^2 + C)}}.$

Then $mt = \log_e \{y + \sqrt{(y^2 + C)}\} + \alpha,$

where α is a constant, or we have

$$\sqrt{(y^2 + C)} + y = e^{mt - \alpha}.$$

Hence $\frac{1}{\sqrt{(y^2 + C)} + y} = e^{\alpha - mt}.$

Multiply the numerator and denominator of the left-hand member by $\sqrt{(y^2 + C)} - y$ and note that

$$\{\sqrt{(y^2 + C)} + y\} \{\sqrt{(y^2 + C)} - y\} = C,$$

we find

$$\frac{\sqrt{(y^2 + C)} - y}{C} = e^{\alpha - mt},$$

or

$$\sqrt{(y^2 + C)} - y = Ce^{\alpha - mt}.$$

But

$$\sqrt{(y^2 + C)} + y = e^{mt - \alpha}.$$

Therefore

$$2y = e^{mt - \alpha} - Ce^{\alpha - mt}.$$

Put

$$\frac{1}{2} e^{-\alpha} = A, \quad -\frac{1}{2} Ce^{\alpha} = B.$$

Then

$$y = Ae^{mt} + Be^{-mt}.$$

If we supposed that $m \frac{dt}{dy} = -\frac{1}{\sqrt{(y^2 + C)}}$ we could obtain the result by writing $-t$ for t . The form of the result is unaltered. We see that every solution of the equation $\frac{d^2y}{dt^2} = m^2y$ must be of the form $y = Ae^{mt} + Be^{-mt}$, where A and B may be any constants.

EXAMPLES

Integrate the following (1)—(4):—

$$(1) \frac{1}{\sqrt{(x^2 + 2x)}}, \quad (2) \frac{1}{\sqrt{(x^2 + x)}}, \quad (3) \frac{1}{\sqrt{(2x^2 + 3)}}, \quad (4) \frac{1}{\sqrt{(3x^2 - 1)}}.$$

INTEGRATION BY PARTS

128. An important method of integration is founded upon the rule for differentiating a product. We have

$$\frac{d(uv)}{dx} = v \frac{du}{dx} + u \frac{dv}{dx},$$

which is the same as

$$uv = \int v \frac{du}{dx} dx + \int u \frac{dv}{dx} dx.$$

We have therefore the equation

$$\int v \frac{du}{dx} dx = uv - \int u \frac{dv}{dx} dx.$$

This formula is known as the rule of "integration by parts."

129. We consider some examples of integration by parts.

$$(i) \int \log_e x dx.$$

Let $u = x$, $v = \log_e x$,

$$\begin{aligned} \text{then} \quad \int \log_e x dx &= x \log_e x - \int x \frac{1}{x} dx \\ &= x \log_e x - x. \end{aligned}$$

$$(ii) \int x^n \log_e x dx.$$

As before it will be convenient to put $v = \log_e x$, because then $\frac{dv}{dx}$ is a rational function, viz. $\frac{1}{x}$. The right form for u is then given by the requirement that x^n should be $\frac{du}{dx}$. We see that we ought to put $u = \frac{x^{n+1}}{n+1}$.

Then

$$\begin{aligned} \int x^n \log_e x dx &= \frac{x^{n+1}}{n+1} \log_e x - \int \frac{x^{n+1}}{n+1} \frac{1}{x} dx \\ &= \frac{x^{n+1}}{n+1} \log_e x - \frac{x^{n+1}}{(n+1)^2}. \end{aligned}$$

This formula holds for all values of n except -1 . For this case see Ex. 5 (6), p. 92.

This example illustrates the choice of functions u , v in an integral which is being attacked by the method of integration by parts. Let $\int \phi(x) \psi(x) dx$ be such an integral, and let $\phi(x)$ be chosen as v , then $\psi(x) = \frac{du}{dx}$, or $u = \int \psi(x) dx$. If we are seeking such an integral as $\int f(x) dx$, where $f(x)$ can be resolved into two factors $\phi(x)$, $\psi(x)$ in many ways, we must choose a way in which one factor $\psi(x)$ is easy to integrate, and we generally try to choose a way in which the other factor $\phi(x)$ shall be very much simplified by differentiation.

$$(iii) \quad \int x e^x dx.$$

Put

$$u = e^x, \quad v = x.$$

Then

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x.$$

$$(iv) \quad \int x^n e^x dx.$$

Put

$$u = e^x, \quad v = x^n.$$

Then

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx. \quad (\alpha)$$

So

$$\int x^{n-1} e^x dx = x^{n-1} e^x - (n-1) \int x^{n-2} e^x dx.$$

The process can be repeated until the required integral is found, n being a positive integer.

A formula like (α) of this example, by which an integral is made to depend upon a simpler integral of a similar form, so that its value can be determined by repeating the same process a certain number of times, is called a "formula of reduction."

$$(v) \quad \int \sqrt{1+x^2} dx.$$

Put

$$u = x, \quad v = \sqrt{1+x^2},$$

then

$$\int \sqrt{1+x^2} dx = x \sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}} dx.$$

But

$$\frac{x^2}{\sqrt{1+x^2}} = \frac{(1+x^2) - 1}{\sqrt{1+x^2}} = \sqrt{1+x^2} - \frac{1}{\sqrt{1+x^2}}.$$

Hence
$$\int \sqrt{1+x^2} dx = x \sqrt{1+x^2} - \int \sqrt{1+x^2} dx + \int \frac{1}{\sqrt{1+x^2}} dx,$$

and therefore
$$2 \int \sqrt{1+x^2} dx = x \sqrt{1+x^2} + \int \frac{1}{\sqrt{1+x^2}} dx$$

$$= x \sqrt{1+x^2} + \log_e \{x + \sqrt{1+x^2}\},$$

or
$$\int \sqrt{1+x^2} dx = \frac{1}{2} x \sqrt{1+x^2} + \frac{1}{2} \log_e \{x + \sqrt{1+x^2}\}.$$

130. We consider some examples of integrals involving trigonometric functions.

(i)
$$\int x \cos x dx.$$

We integrate by parts, using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx,$$

and putting

$$u = x, \quad v = \sin x.$$

We get

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x.$$

(ii)
$$\int x^n \cos x dx,$$

Here we put $u = x^n$, $v = \sin x$ and find

$$\int x^n \cos x dx = x^n \sin x - \int n x^{n-1} \sin x dx.$$

Again in $\int x^{n-1} \sin x dx$ we put $u = x^{n-1}$, $v = \cos x$ and find

$$- \int x^{n-1} \sin x dx = x^{n-1} \cos x - \int (n-1) x^{n-2} \cos x dx.$$

Hence we have

$$\int x^n \cos x dx = x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x dx.$$

This is a formula of reduction (cf. § 129 (iv)).

(iii)
$$\int e^{ax} \cos bx dx.$$

Here we put $v = e^{ax}$, $u = \cos bx$, and find

$$\int a e^{ax} \cos bx dx = e^{ax} \cos bx + \int b e^{ax} \sin bx dx.$$

In like manner in $\int e^{ax} \sin bx$ we put $v = e^{ax}$, $u = \sin bx$, and find

$$\int a e^{ax} \sin bx dx = e^{ax} \sin bx - \int b e^{ax} \cos bx dx.$$

Hence we have

$$\int e^{ax} \cos bx dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} \left\{ e^{ax} \sin bx - b \int e^{ax} \cos bx dx \right\}$$

$$\text{or} \quad \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \cos bx dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx,$$

$$\text{or} \quad \int e^{ax} \cos bx dx = e^{ax} \frac{b \sin bx + a \cos bx}{a^2 + b^2}.$$

EXAMPLES

1. Integrate the following (1)–(11) :—

- | | | | |
|---------------------|-------------------------|---------------------------------|--------------------|
| (1) $\log_{10} x$, | (2) $x^2 \log_{10} x$, | (3) $\frac{\log_{10} x}{x^2}$, | (4) $x^3 e^x$, |
| (5) $x^4 e^x$, | (6) $x e^{-x}$, | (7) $x \sin x$, | (8) $x^2 \cos x$, |
| (9) $x^2 \sin x$, | (10) $x^3 \cos x$, | (11) $x^3 \sin x$. | |

2. Prove that $\int \sqrt{x^2 + A} dx = \frac{1}{2} [x \sqrt{x^2 + A} + A \log_e \{x + \sqrt{x^2 + A}\}]$.

3. Find $\int \sqrt{x^2 + px + q} dx$.

4. Obtain a formula of reduction for $\int e^{-ax} x^n dx$.

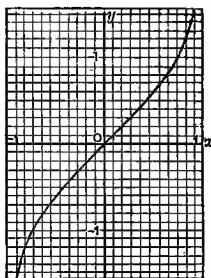
5. Obtain the value of $\int e^{ax} \sin bx dx$ by the method of § 130 (iii).

INVERSE TRIGONOMETRIC FUNCTIONS

131. We can give a different form to some of the results in Ch. VII. Instead of writing y for $\sin \theta$ and x for θ in the table of § 105, we may write x for $\sin \theta$ and y for θ , so that $x = \sin y$. By means of the table we can draw the graph of y as a function of x for values of x lying between 0 and 1; and by means of the relation $\sin(-y) = -\sin y$ we can continue the graph on the left-hand side of the axis of y , and so draw the graph of y as a function of x for values of x lying between -1 and 0. Since the sine of a number always lies between -1 and 1 there

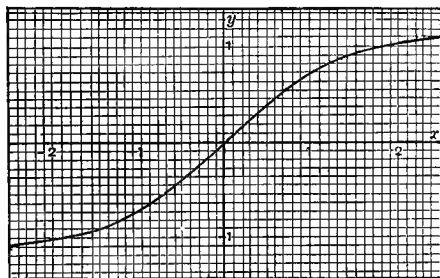
is no value for y as a function of x except when x lies between -1 and 1 . The extreme values of y are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. The equation $x = \sin y$ by which this function is defined is written $y = \sin^{-1} x$.

Fig. 54 is the graph of $\sin^{-1} x$.



$$y = \sin^{-1} x.$$

Fig. 54.



$$y = \tan^{-1} x.$$

Fig. 55.

In like manner by means of the equation $x = \tan y$ we may define a function of x . We write $y = \tan^{-1} x$. The graph of the function $\tan^{-1} x$ is shown in Fig. 55 for values of x lying between -2.5 and 2.5 . There is a definite value of y for every value of x . All the values of $\tan^{-1} x$ lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

The functions $\sin^{-1} x$ and $\tan^{-1} x$ are called the *inverse sine* and *inverse tangent* of x . We could, of course, define in the same way the inverse cosine, inverse secant and so on.

132. We can at once differentiate the inverse sine and inverse tangent.

(i) Let $x = \sin y, \quad y = \sin^{-1} x.$

Then $\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$

Hence $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$

Since the extreme values of y are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, $\cos y$ is a positive number for all the values of y that occur, and the square root must be taken to have the positive sign.

(ii) Let $x = \tan y, \quad y = \tan^{-1} x.$

Then

$$\frac{dx}{dy} = \sec^2 y = \frac{1}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = 1 + \frac{\sin^2 y}{\cos^2 y} = 1 + \tan^2 y = 1 + x^2.$$

Hence $\frac{dy}{dx} = \frac{1}{1 + x^2}.$

These results give us two new formulae of integration, viz.

$$\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x,$$

$$\int \frac{1}{1 + x^2} dx = \tan^{-1} x.$$

By using the rule for differentiating a function of a function (§ 25) we find the results

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \quad (\text{H}),$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad (\text{I}).$$

These are standard forms.

133. The chief purpose of the introduction of the inverse sine and inverse tangent into the elements of the Integral Calculus is the integration of such expressions as $(a^2 - x^2)^{-\frac{1}{2}}$ and $(a^2 + x^2)^{-1}$ and of others which can be reduced to them.

We consider some examples of such reduction.

$$(i) \int \frac{1}{\sqrt{(q+px-x^2)}} dx.$$

By the method of § 124 we may write

$$q+px-x^2 = \left(q + \frac{1}{4}p^2\right) - \left(x - \frac{1}{2}p\right)^2.$$

If $q + \frac{1}{4}p^2$ were negative the whole expression would be negative, and would not have a real square root. We therefore take $q + \frac{1}{4}p^2$ to be positive, and put

$$q + \frac{1}{4}p^2 = \gamma^2, \text{ and } \gamma = \sqrt{\left(q + \frac{1}{4}p^2\right)}.$$

Then, writing
we have

$$x - \frac{1}{2}p = z,$$

$$\int \frac{1}{\sqrt{(q+px-x^2)}} dx = \int \frac{1}{\sqrt{(\gamma^2 - z^2)}} dz = \sin^{-1} \frac{z}{\gamma} = \sin^{-1} \frac{x - \frac{1}{2}p}{\sqrt{\left(q + \frac{1}{4}p^2\right)}}.$$

$$(ii) \int \frac{ax + \beta}{x^2 + px + q} dx.$$

In § 124 we saw how to find this integral if $p^2 > 4q$. We now take $p^2 < 4q$ and write

$$x^2 + px + q = \left(x + \frac{1}{2}p\right)^2 + \left(q - \frac{1}{4}p^2\right).$$

Since $q - \frac{1}{4}p^2$ is positive, we may put

$$q - \frac{1}{4}p^2 = \gamma^2 \text{ and } \gamma = \sqrt{\left(q - \frac{1}{4}p^2\right)},$$

and then, writing

$$x + \frac{1}{2}p = z, \text{ so that } z^2 + \gamma^2 = x^2 + px + q,$$

we have

$$\begin{aligned} \int \frac{ax + \beta}{x^2 + px + q} dx &= \int \frac{az + \left(\beta - \frac{1}{2}ap\right)}{z^2 + \gamma^2} dz \\ &= a \int \frac{z}{z^2 + \gamma^2} dz + \left(\beta - \frac{1}{2}ap\right) \int \frac{1}{z^2 + \gamma^2} dz. \end{aligned}$$

Now,

$$\int \frac{z}{z^2 + \gamma^2} dz = \frac{1}{2} \log_e (z^2 + \gamma^2),$$

and

$$\int \frac{1}{z^2 + \gamma^2} dz = \frac{1}{\gamma} \tan^{-1} \frac{z}{\gamma}.$$

On substituting $x + \frac{1}{2}p$ for z , $\sqrt{\left(q - \frac{1}{4}p^2\right)}$ for γ , and $x^2 + px + q$ for $z^2 + \gamma^2$, we find

$$\int \frac{\alpha x + \beta}{x^2 + px + q} dx = \frac{1}{2} \alpha \log_e (x^2 + px + q) + \frac{\beta - \frac{1}{2} \alpha p}{\sqrt{\left(q - \frac{1}{4} p^2\right)}} \tan^{-1} \frac{x + \frac{1}{2} p}{\sqrt{\left(q - \frac{1}{4} p^2\right)}}.$$

EXAMPLES

1. Prove that, when h is small, $\sin^{-1}(x+h) \doteq \sin^{-1}x + \frac{h}{\sqrt{1-x^2}}$, and that $\tan^{-1}(x+h) \doteq \tan^{-1}x + \frac{h}{1+x^2}$.

2. The sine of an angle is estimated as 0.4. Assuming that the error is not greater than 0.001, find approximately the possible error of the angle in degrees, minutes and seconds. [Result 3' 45'']

3. The tangent of an angle is estimated as 1.5. Assuming that the error is not greater than 0.001, find approximately the possible error of the angle in degrees, minutes and seconds. [Result 1' 3'' 5.]

4. Differentiate $x \sin^{-1} x$, $\frac{1}{x} \tan^{-1} x$.

5. Integrate the following (1)–(7):—

$$\begin{aligned} (1) \frac{1}{\sqrt{1-3x^2}}, \quad (2) \frac{1}{\sqrt{5-2x^2}}, \quad (3) \frac{1}{\sqrt{1+x-x^2}}, \quad (4) \frac{1}{\sqrt{3+2x-x^2}}, \\ (5) \frac{1}{\sqrt{3+x-2x^2}}, \quad (6) \frac{x+1}{x^2+x+1}, \quad (7) \frac{2x+3}{x^2+2x+3}. \end{aligned}$$

134. In many cases it is better to avoid introducing the inverse functions, and simply to use a substitution suggested by the previous results.

For example, consider $\int \frac{1}{\sqrt{(a^2 - x^2)}} dx$. We go back to first principles and observe that we are required to find a function y which shall satisfy the equation

$$\frac{dy}{dx} = \frac{1}{\sqrt{(a^2 - x^2)}}.$$

Now put $x = a \sin \theta$, then $\sqrt{(a^2 - x^2)} = a \cos \theta$,

$$\text{and} \quad \frac{dx}{d\theta} = a \cos \theta,$$

so that we have

$$\frac{dy}{d\theta} = \frac{dy}{dx} \frac{dx}{d\theta} = \frac{1}{a \cos \theta} a \cos \theta = 1,$$

and $y = \theta$ satisfies the equation. That is to say, if we put $x = a \sin \theta$, all the values of $\int \frac{1}{\sqrt{(a^2 - x^2)}} dx$ are included in the formula $\theta + C$, where C is an arbitrary constant.

We might put $x = a \cos \phi$ instead of $x = a \sin \theta$. We should find in the same way $\frac{dy}{d\phi} = -1$, and so $y = C' - \phi$, where C' is an arbitrary constant.

The two forms can be reconciled by observing that if $\frac{1}{2}\pi - \phi = \theta$, then $\cos \phi = \sin \theta$.

135. We consider some examples of such substitutions.

$$(i) \quad \int \sqrt{(a^2 - x^2)} dx.$$

Put $x = a \sin \theta$, the integral becomes $a^2 \int \cos^2 \theta d\theta$. We can find this integral from the result of Ex. 1 (7) on p. 114, or we may find it directly by integration by parts. We use the formula

$$\int u \frac{dv}{d\theta} d\theta = uv - \int v \frac{du}{d\theta} d\theta,$$

putting $u = \cos \theta$, $v = \sin \theta$, and getting

$$\begin{aligned} \int \cos^2 \theta d\theta &= \sin \theta \cos \theta + \int \sin^2 \theta d\theta \\ &= \sin \theta \cos \theta + \int (1 - \cos^2 \theta) d\theta \\ &= \sin \theta \cos \theta + \theta - \int \cos^2 \theta d\theta. \end{aligned}$$

$$\text{Hence} \quad 2 \int \cos^2 \theta d\theta = \sin \theta \cos \theta + \theta,$$

and
$$\int \cos^2 \theta \, d\theta = \frac{1}{2} (\theta + \sin \theta \cos \theta).$$

In like manner we should find

$$\int \sin^2 \theta \, d\theta = \frac{1}{2} (\theta - \sin \theta \cos \theta).$$

We find hence some important definite integrals, viz.

$$\int_0^{\frac{1}{2}\pi} \sin^2 \theta \, d\theta = \int_0^{\frac{1}{2}\pi} \cos^2 \theta \, d\theta = \frac{1}{4} \pi.$$

We may also find the definite integral $\int_0^a \sqrt{(a^2 - x^2)} \, dx$. The indefinite integral is $\frac{1}{2} \left\{ a^2 \sin^{-1} \frac{x}{a} + x \sqrt{(a^2 - x^2)} \right\}$, and, since this vanishes when $x=0$, no constant need be added. As x increases from 0 to a , $\sin^{-1} \frac{x}{a}$ increases from 0 to $\frac{1}{2} \pi$. Hence

$$\int_0^a \sqrt{(a^2 - x^2)} \, dx = \frac{1}{4} \pi a^2.$$

$$(ii) \quad \int x^2 \sqrt{(a^2 - x^2)} \, dx.$$

Put $x = a \sin \theta$, we find

$$\begin{aligned} \int x^2 \sqrt{(a^2 - x^2)} \, dx &= a^4 \int \sin^2 \theta \cos^2 \theta \, d\theta, \\ &= a^4 \int (\cos^2 \theta - \cos^4 \theta) \, d\theta. \end{aligned}$$

To find $\int \cos^4 \theta \, d\theta$ we use the formula of integration by parts in the form written in Ex. (i) putting $u = \cos^3 \theta$, $v = \sin \theta$, and getting,

$$\begin{aligned} \int \cos^4 \theta \, d\theta &= \cos^3 \theta \sin \theta + \int 3 \cos^2 \theta \sin^2 \theta \, d\theta. \\ &= \cos^3 \theta \sin \theta + 3 \int (\cos^2 \theta - \cos^4 \theta) \, d\theta \\ &= \cos^3 \theta \sin \theta + \frac{3}{2} (\theta + \sin \theta \cos \theta) - 3 \int \cos^4 \theta \, d\theta. \end{aligned}$$

Hence
$$4 \int \cos^4 \theta \, d\theta = \cos^3 \theta \sin \theta + \frac{3}{2} (\theta + \sin \theta \cos \theta),$$

and
$$\int \cos^4 \theta \, d\theta = \frac{1}{4} \cos^3 \theta \sin \theta + \frac{3}{8} (\theta + \sin \theta \cos \theta).$$

We now find

$$\begin{aligned}\int x^2 \sqrt{(a^2 - x^2)} dx &= a^4 \left[\frac{1}{2} (\theta + \sin \theta \cos \theta) \right. \\ &\quad \left. - \left\{ \frac{1}{4} \cos^3 \theta \sin \theta + \frac{3}{8} (\theta + \sin \theta \cos \theta) \right\} \right] \\ &= a^4 \left(\frac{1}{8} \theta + \frac{1}{8} \sin \theta \cos \theta - \frac{1}{4} \sin \theta \cos^3 \theta \right).\end{aligned}$$

From the above work we deduce the important definite integral

$$\int_0^{\frac{1}{2}\pi} \cos^4 \theta d\theta = \frac{3}{16} \pi.$$

We may also find the definite integral $\int_{-a}^a x^2 \sqrt{(a^2 - x^2)} dx$. The in-

definite integral is $\frac{1}{8} a^4 \sin^{-1} \frac{x}{a} + \frac{1}{8} a^2 x \sqrt{(a^2 - x^2)} - \frac{1}{4} x (a^2 - x^2)^{\frac{3}{2}}$. We add a constant **C** and adjust **C** so that the integral may vanish when $x = -a$.

When $x = -a$, $\sin^{-1} \frac{x}{a} = -\frac{1}{2} \pi$, and therefore $\mathbf{C} = \frac{1}{16} \pi a^4$. As x increases from

$-a$ to a , $\sin^{-1} \frac{x}{a}$ increases from $-\frac{1}{2} \pi$ to $\frac{1}{2} \pi$, and when $x = a$ the value of the

indefinite integral written above is $\frac{1}{16} \pi a^4$. Hence

$$\int_{-a}^a x^2 \sqrt{(a^2 - x^2)} dx = \frac{1}{16} \pi a^4 + \mathbf{C} = \frac{1}{8} \pi a^4.$$

$$(iii) \quad \int \frac{1}{\cos \theta} d\theta.$$

We know that $\int \frac{1}{\sqrt{(1+x^2)}} dx = \log_e \{x + \sqrt{(1+x^2)}\}$.

Now put $x = \tan \theta$.

We find $\sqrt{(1+x^2)} = \sec \theta$, $\frac{dx}{d\theta} = \sec^2 \theta$,

and therefore $\int \frac{1}{\sqrt{(1+x^2)}} dx = \int \sec \theta d\theta = \int \frac{1}{\cos \theta} d\theta$.

We have therefore the result

$$\int \frac{1}{\cos \theta} d\theta = \log_e (\tan \theta + \sec \theta),$$

or, as it may be written,

$$\int \frac{1}{\cos \theta} d\theta = \log_e \frac{1 + \sin \theta}{\cos \theta}.$$

EXAMPLES

1. Integrate the following (1)–(5):—

$$(1) \sin^2 x \cos x, \quad (2) \cos^2 x \sin x, \quad (3) \sin^3 x,$$

$$(4) \cos^3 x, \quad (5) \sin^4 x.$$

2. Prove that
$$\int_0^{\frac{1}{2}\pi} \sin^4 x \, dx = \frac{3}{16} \pi.$$

3. Prove that
$$\int \frac{1}{\sin x} \, dx = \log_e \frac{1 - \cos x}{\sin x}.$$

ADDITIONAL EXERCISES

[It is not intended that these examples should all be worked before the rest of the book is read. The student will find it profitable to work a few of these every day for the sake of practice.]

1. Integrate with respect to x the following (1)—(50) :

- (1) $\frac{1}{6-x^2}$, (2) $\frac{x}{6-2x^2}$, (3) $\frac{3x+1}{6-3x^2}$,
- (4) $\frac{1}{5+x^2}$, (5) $\frac{x}{5+2x^2}$, (6) $\frac{3x+3}{5+3x^2}$,
- (7) $\frac{1}{x^2+4x+1}$, (8) $\frac{x}{x^2+4x+2}$, (9) $\frac{2x-1}{x^2+4x+3}$,
- (10) $\frac{1}{x^2+4x+5}$, (11) $\frac{x}{x^2+4x+6}$, (12) $\frac{2x+1}{x^2+4x+7}$,
- (13) $\frac{1}{(x-1)(x-2)(x-3)}$, (14) $\frac{x}{(x-1)(x-2)(3-x)}$,
- (15) $\frac{x^2}{(x-1)(2-x)(3-x)}$,
- (16) $\frac{1}{\sqrt{(2x^2-5)}}$, (17) $\frac{x}{\sqrt{(2x^2-6)}}$, (18) $\frac{1}{\sqrt{(6-2x^2)}}$, (19) $\frac{x}{\sqrt{(5-2x^2)}}$,
- (20) $\sqrt{(2x^2-3)}$, (21) $x\sqrt{(2x^2+3)}$, (22) $\sqrt{(2-3x^2)}$,
- (23) $x\sqrt{(1-3x^2)}$, (24) $x^2\sqrt{(2x^2+5)}$, (25) $x^2\sqrt{(3-2x^2)}$,
- (26) $(x^2+2x+3)\sqrt{(1+x)}$, (27) $(x^2-3x+2)\sqrt{(1-x)}$,
- (28) $x\sqrt{(2x+x^2)}$, (29) $(x-1)\sqrt{(2x-x^2)}$,
- (30) $\frac{1}{\sqrt{(2x^2+2x-1)}}$, (31) $\frac{x}{\sqrt{(2x^2-2x-1)}}$,
- (32) $\frac{1}{\sqrt{(2+2x-x^2)}}$, (33) $\frac{x}{\sqrt{(1+x-2x^2)}}$,
- (34) $\sqrt{(x^2+x-1)}$, (35) $x\sqrt{(x^2+x+1)}$, (36) $\sqrt{(1+x-x^2)}$,
- (37) $x\sqrt{(1-x-x^2)}$, (38) $\sin^{-1} x$ (39) $x \sin^{-1} x$,
- (40) $x^2 \sin^{-1} x$, (41) $x^3 \sin^{-1} x$, (42) $\tan^{-1} x$,
- (43) $x \tan^{-1} x$, (44) $x^2 \tan^{-1} x$, (45) $x^3 \tan^{-1} x$, (46) $e^x \cos^2 x$,
- (47) $e^x \sin^2 x$, (48) $e^{-x} \cos^2 x$, (49) $e^{-x} \sin^2 x$,
- (50) $(e^x + e^{-x}) \sin x \cos x$.

2. Compute approximate values of the following definite integrals (1)–(10):—

- $$\begin{aligned}
 (1) \int_3^4 \frac{1}{(x^2-4)} dx, \quad (2) \int_2^3 \frac{1}{(x-1)(4-x)} dx, \quad (3) \int_1^2 \frac{1}{\sqrt{(x^2-1)}} dx, \\
 (4) \int_0^1 \frac{1}{\sqrt{(3x^2+2)}} dx, \quad (5) \int_1^2 \sqrt{(x^2-1)} dx, \quad (6) \int_0^1 \frac{1}{\sqrt{(2-x^2)}} dx, \\
 (7) \int_3^4 \frac{1}{\sqrt{(25-x^2)}} dx, \quad (8) \int_1^2 \frac{1}{\sqrt{(5-x^2)}} dx, \quad (9) \int_3^4 \sqrt{(36-x^2)} dx, \\
 (10) \int_0^{\frac{1}{2}\pi} \frac{1}{\cos x} dx.
 \end{aligned}$$

[The results, correct to 3 places of decimals, are (1) 0.128, (2) 0.462, (3) 1.317, (4) 0.569, (5) 1.074, (6) 0.785, (7) 0.284, (8) 0.646, (9) 13.803, (10) 0.881. In (7), (8), (9) the functions $\sin^{-1} \frac{x}{5}$, $\sin^{-1} \frac{x}{\sqrt{5}}$, $\sin^{-1} \frac{x}{6}$ occur, and their values are to be computed for some particular values of x . To compute roughly the value of such an expression as $\sin^{-1} \frac{2}{\sqrt{5}}$, we find first that $\frac{2}{\sqrt{5}} \doteq 0.8944272$, then from a book of Tables that this is slightly greater than $\sin 63^\circ 26'$, then that the radian measure of the angle $63^\circ 26'$ is 1.1071..., and that of the angle $63^\circ 27'$ is 1.1074.... Thus we find $\sin^{-1} \frac{2}{\sqrt{5}} \doteq 1.107$.]

CHAPTER IX

VARIOUS RESULTS CONNECTED WITH ARCS OF CURVES

136. IN order to obtain formulae for measuring the lengths of arcs of curves we extend to curves in general a result which we used in the case of a circle, *viz.* if the length of an arc PQ is l units of length, and the length of the chord PQ is χ units of length, then, as Q moves up to P, $\frac{l}{\chi}$ tends to 1 as a limit.

If the length of the arc AP of a curve is s units of length, the length of the arc AQ is $s + \Delta s$ units of length, and the length of the chord PQ is χ units of length, $\frac{\Delta s}{\chi}$ tends to 1 as a limit when Δs tends to zero, provided the points A, P, Q are so situated that Δs is positive (Fig. 56). If Δs is negative the limit is -1 .

If x, y are the coordinates of P, $x + \Delta x$ and $y + \Delta y$ those of Q, we have at once

$$\chi^2 = (\Delta x)^2 + (\Delta y)^2,$$

and therefore

$$\frac{ds}{dx} = \pm \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}, \quad \frac{ds}{dy} = \pm \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}},$$

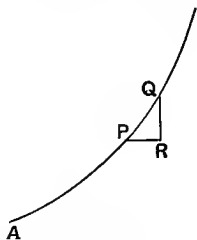


Fig. 56.

the sign to be taken + or - according as x , or y , increases or decreases as s increases.

Exactly as in § 109 we may define an angle ϕ by drawing Px' to the right from P parallel to the axis of x , and the tangent PT in the sense of increase of s . The angle $x'PT$ traced out by a line revolving (in the counterclockwise sense) from the position Px' into the position PT is ϕ radians (Fig. 57).

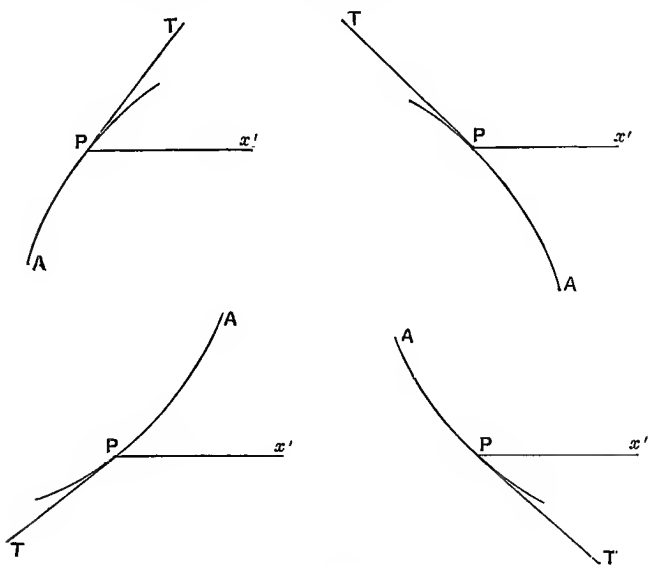


Fig. 57.

In all the figures we have

$$\frac{dx}{ds} = \cos \phi, \quad \frac{dy}{ds} = \sin \phi.$$

We have also in general

$$\frac{dy}{dx} = \tan \phi \quad \text{and} \quad \frac{dx}{dy} = \cot \phi.$$

The first of these holds if ϕ is not an odd multiple of $\frac{1}{2}\pi$, the second if ϕ is not zero or a multiple of π .

137. The last result shows that the gradient of a curve at a point, or the gradient of the tangent to the curve at the point, is the (trigonometrical) tangent of the angle which the (geometrical) tangent to the curve at the point makes with the axis of x . If the (geometrical) tangent meets the axis of x in a point, say T, the angle in question is supposed to be generated by a straight line drawn from T along the axis of x to the right, and revolving in the counterclockwise sense from this position into the position of the tangent. This statement is necessary in order to secure that the (trigonometrical) tangent has the same sign as the gradient.

138. We apply the formula

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

to find the length of the arc AP of a circle given by the equation $x^2 + y^2 = r^2$, where P is a point (x, y) in the first quadrant as shown in Fig. 58. The sign has been taken to be +, and this is shown by the figure to be right.

We have

$$x^2 = r^2 - y^2, \quad x = \sqrt{r^2 - y^2},$$

$$\frac{dx}{dy} = -\frac{y}{\sqrt{r^2 - y^2}}, \quad 1 + \left(\frac{dx}{dy}\right)^2 = \frac{r^2}{r^2 - y^2}.$$

Hence

$$\frac{ds}{dy} = \frac{r}{\sqrt{r^2 - y^2}},$$

and therefore

$$s = r \int \frac{1}{\sqrt{r^2 - y^2}} dy + C = r \sin^{-1} \frac{y}{r} + C.$$

Also, when $y=0$, $s=0$, and therefore

$$s = r \sin^{-1} \frac{y}{r}.$$

If the angle AOP is θ radians, y is $r \sin \theta$, and our result is the same as the known formula

$$s = r\theta.$$

139. As an additional example we find the length of an arc of a parabola. Let the equation to the parabola be $y = x^2$, and let us find the length of

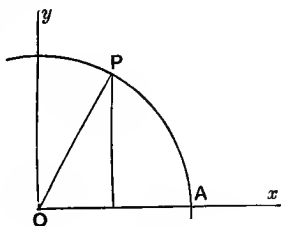


Fig. 58.

the arc contained between the origin and any point (x, y) on the curve. We may take s and x to increase together.

We have $y = x^2, \quad \frac{dy}{dx} = 2x, \quad \frac{ds}{dx} = \sqrt{1+4x^2},$

and $s=0$ when $x=0$.

Hence
$$s = \int \sqrt{1+4x^2} \, dx + C$$

$$= \frac{1}{2} x \sqrt{1+4x^2} + \frac{1}{4} \log_e \{2x + \sqrt{1+4x^2}\} + C,$$

where C is determined by putting s and x both equal to 0. We find $C=0$, and therefore

$$s = \frac{1}{2} x \sqrt{1+4x^2} + \frac{1}{4} \log_e \{2x + \sqrt{1+4x^2}\}.$$

CURVATURE

140. Let AB be an arc of a curve, P a point on this arc. We can think of P as travelling along the curve from A to B . From a point O draw a straight line Op parallel to the tangent at P , drawn in the sense of description of the curve. We can take

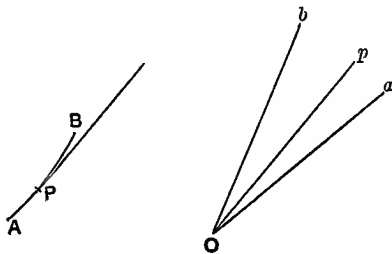


Fig. 59.

the arc AB to be so short that (i) Op turns about O always in the same sense, (ii) the angle through which it turns between its extreme positions Oa, Ob is acute (Fig. 59). Let the length of the arc AB be l units of length, and the angle aOb be γ radians. The angle γ radians is the angle through which the tangent turns

as P moves from A to B . It may properly be called the "total curvature" of the arc AB . The quotient $\frac{\gamma}{l}$ measures, in radians per unit of length, a quantity which may properly be called the "average curvature" of the arc AB . The limit κ to which this quotient tends as B is brought nearer and nearer to A is defined to be the measure, in radians per unit of length, of the "curvature" of the curve at the point A .

141. If the curve is a circle (Fig. 60), the angle which the arc AB subtends at the centre is γ radians, and, if the radius is r units of length, $l = \gamma r$. Therefore $\kappa = \frac{1}{r}$. Therefore the curvature of a circle of radius r units of length is the same at every point, and equal to $\frac{1}{r}$ radians per unit of length.

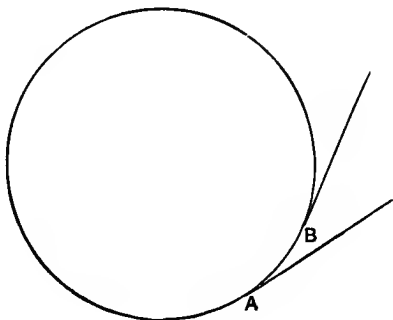


Fig. 60.

142. In the case of any curve the length $\frac{1}{\kappa}$ units of length is the "radius of curvature" of the curve at the point A . We write ρ for $\frac{1}{\kappa}$.

On the normal at A take a point C so that (i) the part of the curve near to A is concave to C, (ii) the length of AC is ρ units of length. With centre C and radius ρ units of length describe a circle. The point C is called the "centre of curvature" of the curve at A, and the circle is called the "circle of curvature" of the curve at A.

143. Let s and ϕ be defined as in § 136. Then ϕ may increase as s increases, or it may diminish. When s becomes $s + \Delta s$, and ϕ becomes $\phi + \Delta\phi$, the absolute value of Δs (sign disregarded) is the number of units of length in the length of an arc, and the absolute value of $\Delta\phi$ is the number of radians in the total curvature of this arc. Hence the absolute value of $\frac{d\phi}{ds}$ is κ or $\frac{1}{\rho}$.

144. When the curve is given by an equation of the form $y = f(x)$, we can find a formula for ρ . We have

$$\frac{dy}{dx} = \tan \phi, \quad \frac{dx}{ds} = \cos \phi.$$

Hence
$$\frac{d^2y}{dx^2} = \sec^2 \phi \frac{d\phi}{dx}, \quad \frac{d\phi}{ds} = \frac{d\phi}{dx} \cos \phi,$$

and therefore
$$\frac{d\phi}{ds} = \cos^3 \phi \frac{d^2y}{dx^2}.$$

The absolute value of $\cos \phi$ is $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}}$, and therefore

$$\frac{1}{\rho} = \pm \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{3}{2}} \frac{d^2y}{dx^2},$$

where the + sign is to be taken when $\frac{d^2y}{dx^2}$ is positive, the - sign

when $\frac{d^2y}{dx^2}$ is negative.

145. As an example consider the parabola given by the equation $y = x^2$. We have

$$\frac{dy}{dx} = 2x, \quad \frac{d^2y}{dx^2} = 2,$$

and therefore

$$\rho = \frac{1}{2} (1 + 4x^2)^{\frac{3}{2}}.$$

146. If a curve differs very little from a straight line, we may take the straight line to be the axis of x . Then y and $\frac{dy}{dx}$ are very small at every point of the curve, and we have the approximate equation

$$\frac{1}{\rho} \approx \pm \frac{d^2y}{dx^2},$$

where the upper or lower sign is to be taken according as $\frac{d^2y}{dx^2}$ is positive or negative.

For example if the equation to the curve is $y = a \sin nx$, where a is a very small constant, $\frac{1}{\rho}$ is approximately equal to the absolute value of $an^2 \sin nx$ or n^2y (sign disregarded).

AREA OF A SURFACE OF REVOLUTION

147. We find the area of the surface of a cone. Let the cone be generated by a right-angled triangle ACB revolving round the side AC (Fig. 61). Let the side AB contain l units of length, and the side BC , r units of length, and let the area of the curved surface be S units of area.

The base of the cone is a circle with C for centre, and its radius is r units of length. In this circle we suppose inscribed a polygon of a large number of sides, and join the vertices P, Q, \dots of the polygon to the vertex A of the cone. We thus form a pyramid on a polygonal base. Let the length of the perpendicular let fall from A upon any side PQ of the polygon be p units of length. If the length of PQ is s units of length, the area of the triangle APQ is $\frac{1}{2} ps$ units of area.

Now let the number of vertices of the polygon be increased, and the lengths of all the sides PQ be diminished indefinitely. The pyramid tends to coincidence with the cone. The sum of the lengths of all the sides of the polygon tends to a limit, which

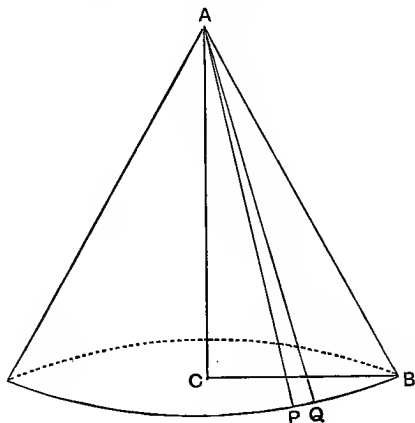


Fig. 61.

is $2\pi r$ units of length, the number p tends to a limit, which is l ; and therefore the sum of the areas of the triangular faces of the pyramid tends to a limit, which is $\frac{1}{2}(2\pi r)l$ units of area. Hence we have

$$S = \pi r l.$$

If α denotes the numerical measure of the angle BAC (so that 2α is the measure of the vertical angle) we have $\frac{r}{l} = \sin \alpha$, and $S = \pi l^2 \sin \alpha$.

Let B be displaced along a generating line of the cone to B' (Fig. 62), so that l becomes $l + \Delta l$, and r becomes $r + \Delta r$. Then α is unaltered, and S becomes $S + \Delta S$, where

$$\Delta S = 2\pi \left(l + \frac{1}{2} \Delta l \right) \sin \alpha \cdot \Delta l,$$

or

$$\Delta S = 2\pi \left(r + \frac{1}{2} \Delta r \right) \Delta l.$$

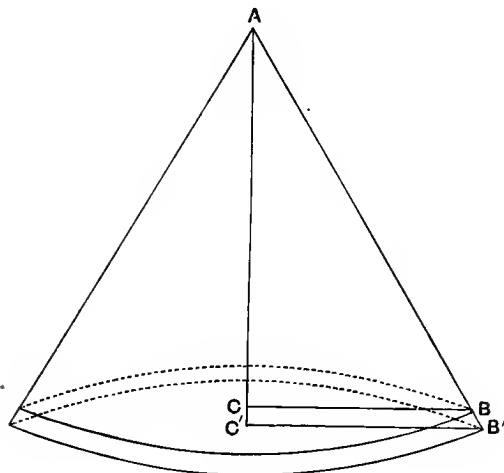


Fig. 62.

148. We take next the question of the area of a belt of a sphere contained between two parallel planes. We think of the sphere as generated by a semicircle revolving about its base. We take the axis of x along the base of the semicircle as in

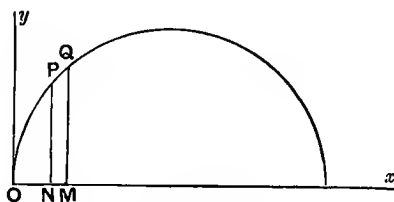


Fig. 63.

Fig. 63. We take a point P on the semicircle, and draw the ordinate PN . Let x, y be the coordinates of P . As the semicircle revolves, the point P traces out a circle whose centre is N and whose radius is y units of length. Let the area of the portion

of the sphere, which contains the point O and is cut off by this circle, be S units of area, and let the length of the arc OP be s units of length. When P moves to Q , so that s becomes $s + \Delta s$, S becomes $S + \Delta S$.

As the semicircle revolves, the arc PQ , of length Δs units of length, traces out a belt of the sphere, and its area is ΔS units of area. The chord PQ traces out a belt of a cone. Let the length of the chord PQ be χ units of length, and let the area of the belt of the cone in question be σ units of area. We know that, as Δs tends to zero, $\frac{\sigma}{\chi}$ tends to a limit, which is $2\pi y$. Now $\frac{\Delta s}{\chi}$ tends to 1 as a limit, and so also does $\frac{\Delta S}{\sigma}$. Hence $\frac{\Delta S}{\Delta s}$ tends to a limit, which is $2\pi y$, or we have

$$\frac{dS}{ds} = 2\pi y.$$

Let the radius of the sphere be a units of length. Then Fig. 64 shows that in every position of P

$$\frac{dx}{ds} = \cos x'PT = \sin PCN = \frac{y}{a},$$

and therefore
$$\frac{dS}{dx} = 2\pi a.$$

Since $S = 0$ when $x = 0$, we have $S = 2\pi ax$.

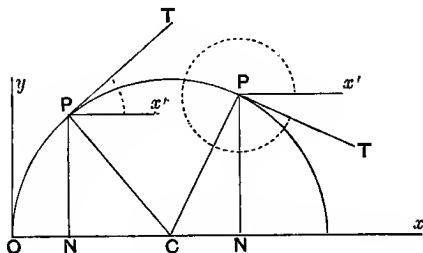


Fig. 64.

It follows that the area of the belt of the sphere contained between two parallel planes, at a distance apart equal to d units of length, is $2\pi ad$ units of area. The area of the whole sphere is $4\pi a^2$ units of area.

149. The reasoning by which we established the equation

$$\frac{dS}{ds} = 2\pi y$$

does not depend upon the revolving curve being a semicircle. It may be applied to the surface traced out by the revolution of any curve about the axis of x .

EXAMPLES

1. The radius of the circular base of a cone is r units of length, and the vertical angle of the cone is $2a$ radians. Express the area of the curved surface in terms of r and a .

2. A piece of paper in the form of a quadrant of a circle is wrapped round a cone, so that the centre of the circle is at the vertex of the cone, and the two bounding semidiameters of the quadrant are seen to lie along the same generating line of the cone. Find the vertical angle of the cone. [Result: $28^\circ 57'$ approximately.]

3. How much cloth is required to cover a lawn-tennis ball of diameter $2\frac{1}{2}$ inches? [Result: 19.63 square inches approximately.]

4. Let the parabola whose equation is $y = \sqrt{x}$ revolve about the axis of x (Fig. 65). Prove with the notation of §§ 148,

149 that $\frac{dS}{dx} = \pi\sqrt{1+4x}$. Hence show

that if the length of ON is 2 units of length, the area of the curved surface traced out by the revolution of the arc OP is 13.6136 units of area approximately.

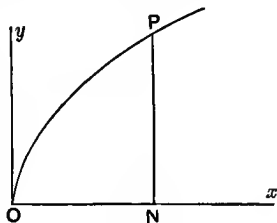


Fig. 65.

CHAPTER X

THE DEFINITE INTEGRAL AS THE LIMIT OF A SUM

150. THE object of such investigations as that in § 71 of the area under a curve is not so much the mensuration of plane figures as a graphic representation of integrals. The number of units of area z in the area bounded above by the graph of $y=f(x)$, below by the axis of x , to the right by a moving ordinate (x) and to the left by a fixed ordinate (a) is expressed as a function of x by the sum of the integral $\int f(x) dx$ and a constant C ; and this constant is determined by the condition that $z=0$ when $x=a$. The value of z when $x=b$ is expressed by the definite integral $\int_a^b f(x) dx$. We may think of the indefinite integral as a definite integral with a variable upper limit, and we may represent it graphically by an area.

In general the only way of finding either the definite integral or the area is to find the indefinite integral by the methods of the Integral Calculus. But if we could find the area by any other method, we should find the value of the definite integral without first finding the indefinite integral. If the indefinite integral is one that we cannot find, we may be able to calculate the area approximately, and then we shall know the integral approximately.

151. The most obvious way of approximating to the area is to draw the graph on squared paper and count the squares that are inside it. The area will be a little greater than the sum of the areas of the included squares. We can improve the approximation very much by quite simple means.

(i) Instead of merely counting the squares complete the rectangles such as PNML (Fig. 66). The sum of the areas of these rectangles is a better approximation to the area than the sum of the areas of the included squares. Let

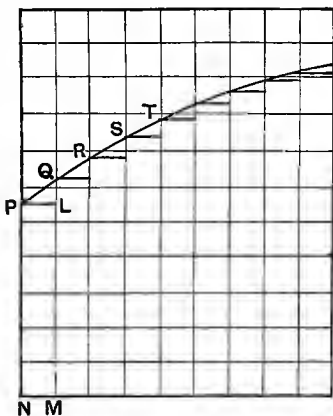


Fig. 66.

the unit of length be such that the distance between consecutive ruled lines is h units of length. If the figure is drawn on a large scale h is a small fraction. Let y_1, y_2, \dots, y_n be the y -coordinates of the first n points. The corresponding sum of areas is

$$h(y_1 + y_2 + \dots + y_{n-1})$$

units of area.

(ii) Instead of completing the rectangles such as PLMN complete the trapeziums such as PQMN. The area of the trapezium between the r th and $(r+1)$ th ordinates is

$$\frac{1}{2} h (y_r + y_{r+1})$$

units of area, and the sum of the areas of the trapeziums gives a good approximation to the required area. The approximate expression for the number of units of area is

$$\frac{1}{2} h \{(y_1 + y_2) + (y_2 + y_3) + \dots + (y_{n-2} + y_{n-1}) + (y_{n-1} + y_n)\}$$

or
$$\frac{1}{2} h \{y_1 + y_n + 2(y_2 + y_3 + \dots + y_{n-1})\}.$$

(iii) Instead of joining two such points as P, Q by a straight line, take the points in threes, P, Q, R, then R, S, T, and so on, and suppose each set of three to lie on a parabola whose axis is parallel to the axis of y (§ 35). This will give a still better approximation. Now we know (§ 73) that the area between the first and third ordinates, the axis of x and the corresponding parabola, is

$$\frac{1}{3} h (y_1 + y_3 + 4y_2),$$

and a like formula holds for all the corresponding pieces of area. Let there be $2n + 1$ points such as P, Q, ..., then the approximate formula for the number of units of area is

$$\frac{1}{3} h \{ (y_1 + y_3 + 4y_2) + (y_3 + y_5 + 4y_4) + \dots + (y_{2n-1} + y_{2n+1} + 4y_{2n}) \},$$

or

$$\frac{1}{3} h \{ y_1 + y_{2n+1} + 2 (y_3 + y_5 + \dots + y_{2n-1}) + 4 (y_2 + y_4 + \dots + y_{2n}) \}.$$

Inside the bracket we have (a) the sum of the first and last y 's, (b) twice the sum of all the other y 's with odd suffixes, (c) four times the sum of all the y 's with even suffixes.

This rule for calculating areas is known as Simpson's Rule.

152. We may use Simpson's Rule to approximate to π . Take a circle whose radius is 1 unit of length, draw the axes of x and y as in Fig. 67, mark the points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$, which are C and D, draw the ordi-

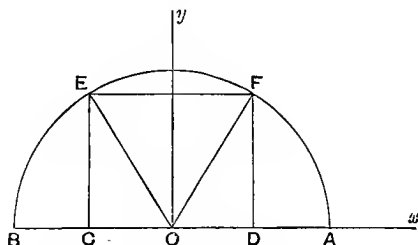


Fig. 67.

nates CE , DF . Then $EF = CD = OA$, and the triangle EOF is equilateral, so that the angle EOF is $\frac{1}{3}\pi$ radians. Also the length of EC or DF is $\sqrt{1 - \left(\frac{1}{2}\right)^2}$ or $\frac{1}{2}\sqrt{3}$ units of length. The area of the curvilinear figure bounded by the arc EF , the ordinates EC , FD and the axis of x is the sum of the areas of the sector EOF , and the triangles EOC , FOD , or it is

$$\frac{1}{2}\frac{\pi}{3} + 2\left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right) \text{ units of area.}$$

To evaluate the same area by Simpson's Rule, divide CD into ten equal pieces. We have $y = \sqrt{1 - x^2}$, $x_1 = -0.5$, $x_2 = -0.4$, ... $x_{11} = 0.5$. We find (Ex. 16, p. 15) for $\frac{1}{3}h\{y_1 + y_{11} + 2(y_3 + y_5 + y_7 + y_9) + 4(y_2 + y_4 + y_6 + y_8 + y_{10})\}$ the value 0.95661 to five decimal places, and we find $\frac{1}{4}\sqrt{3} = 0.43301$ to five decimal places. Hence

$$\frac{1}{6}\pi = 0.52360 \text{ to five decimal places,}$$

and

$$\pi = 3.1416 \text{ to four decimal places.}$$

153. In like manner we may use Simpson's Rule to approximate to $\log_{10} e$. We consider the area contained between the hyperbola whose equation is $y = \frac{1}{x}$ (Fig. 68), the ordinates whose equations are $x=1$ and $x=2$, and the axis of x . We know that this is $\log_e 2$ units of area. We divide the portion of the axis of x contained between the two extreme ordinates into ten equal pieces, so that $y_1 = 1$, $y_2 = \frac{1}{1.1}$, ... $y_{11} = \frac{1}{2}$, $h = 0.1$ and calculate

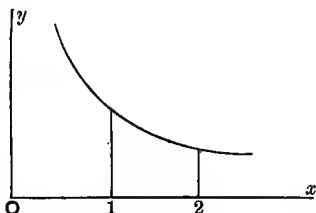


Fig. 68.

$$\frac{1}{3}h[y_1 + y_{11} + 2(y_3 + y_5 + y_7 + y_9) + 4(y_2 + y_4 + y_6 + y_8 + y_{10})].$$

We find (Ex. 17, p. 15) the result 0.69315. Now $\log_{10} 2 = 0.30103$ and $\log_{10} e = \frac{30103}{69315} = 0.43429$ to 5 places. This result gives $e = 2.7183$ to 4 places.

154. We return to the method of approximating to the area by summing the areas of rectangles, and observe that the

approximation may be improved almost as much as may be wished by taking more numerous intermediate ordinates at shorter distances apart. The development of this remark leads to an important result.

To fix ideas we take $f(x)$ to be positive and to increase as x increases. Let A and B be the points of the graph of $y = f(x)$

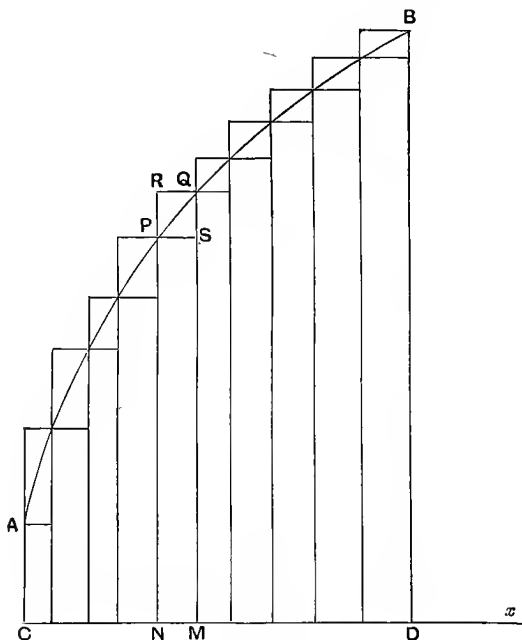


Fig. 69.

at which $x = a$ and $x = b$, and let AC, BD be the ordinates of these points (Fig. 69). Between C and D take a number of intermediate points such as N, M, draw the ordinates such as NP, MQ, and complete the rectangles PNMS, QMNR. The area contained

between the arc PQ , the ordinates PN , QM , and the axis of x is intermediate between the areas of these two rectangles. The sum of the areas of all the "exterior" rectangles, such as $QMNR$, exceeds the sum of the areas of all the "interior" rectangles, such as $PNMS$, by the area of a rectangle whose height is the difference of the extreme ordinates BD , AC and whose base is less than the greatest of the segments such as NM . As the number of the intermediate points such as N is increased, and the lengths of all the segments such as NM are diminished, the difference between the sum of the areas of all the exterior rectangles and the sum of the areas of all the interior rectangles continually diminishes, and can be made as small as we please. The common limit of both these sums of areas is the area of the curvilinear figure $ACDB$.

Now this result means that if we write x_0 for a and x_n for b , and take $n - 1$ intermediate values x_1, x_2, \dots, x_{n-1} , the sum

$$f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \dots + f(x_{k-1})(x_k - x_{k-1}) + \dots \\ + f(x_{n-1})(x_n - x_{n-1})$$

is an approximation to the definite integral $\int_a^b f(x) dx$, and so also is the sum

$$f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + \dots + f(x_k)(x_k - x_{k-1}) + \dots \\ + f(x_n)(x_n - x_{n-1});$$

and, further, that if we invent any rule for increasing the number $n - 1$ of the intermediate values and spacing them out, so that x_0 shall always be a and x_n shall always be b , and all the differences $x_k - x_{k-1}$ shall tend to zero, both these sums tend to the same limit, and this limit is $\int_a^b f(x) dx$. Further, if x'_r denotes a value of x lying between x_{r-1} and x_r , the sum

$$f(x'_1)(x_1 - x_0) + f(x'_2)(x_2 - x_1) + \dots + f(x'_k)(x_k - x_{k-1}) \\ + \dots + f(x'_n)(x_n - x_{n-1})$$

tends to the same limit, viz.: $\int_a^b f(x) dx$. Such a sum may be written

$$\sum_a^b f(x) \Delta x,$$

where the symbol Σ (sigma) means "the sum of such terms as," the letters written above and below indicate the first and last values of x , and Δx denotes in a general way the difference between two consecutive values of x in a series of values placed between a and b .

155. The result would not be altered if $f(x)$ were to diminish as x increases, or if it were to increase in some parts of the range indicated by a , b and diminish in other parts. Nor is it necessary that $f(x)$ should always be positive. The summations for the ranges in which it is positive and those in which it is negative could be performed separately.

Such limits of sums present themselves naturally not only in problems concerned with areas but also in many problems that have nothing to do with areas. The result that such limits can be evaluated as definite integrals enables us to find them whenever we can find the corresponding indefinite integrals.

156. In the case of the area of a figure (§ 75) we may draw across the figure a number of straight lines parallel to the axis of y . If Y units of length is the length intercepted by the figure upon one of these lines, x the x -coordinate of any point on that line, Δx units of length the breadth of a strip between two consecutive lines of the set, the area of the strip of the figure between the two lines can be replaced by the area of a rectangle approximately equal to $Y\Delta x$ units of area. The exact number is expressible as $Y'\Delta x$, where Y' tends to Y as a limit when Δx tends to zero. The limit of the sum $\Sigma Y\Delta x$, where the summation refers to all the strips is the number of units of area in the area of the figure. This result is equivalent to the one found in § 75.

In the case of the volume of a solid (§ 79) we may think of the solid as divided into a large number of slabs or laminae by planes drawn parallel to each other. If Z units of area is the area of the section of the solid by one

of the planes, Δx units of length the distance of this plane from the next, the volume of the lamina is approximately equal to $Z\Delta x$ units of volume. The exact number is expressible as $Z'\Delta x$, where Z' tends to Z as a limit when Δx tends to zero. The limit of the sum $\Sigma Z\Delta x$, where the summation refers to all the laminas, is the number of units of volume of the solid. This result is equivalent to that found in § 79.

In the case of the length of an arc AB of a curve we may begin by marking on the curve a number $n-1$ of points P_1, P_2, \dots, P_{n-1} between the two points A and B. The length, χ_r units of length, of the chord $P_{r-1}P_r$ is given by the equation

$$\chi_r^2 = (x_r - x_{r-1})^2 + (y_r - y_{r-1})^2,$$

where x_r, y_r are the coordinates of P_r . This is equivalent to the general formula

$$\chi^2 = (\Delta x)^2 + (\Delta y)^2$$

of § 136. The number of units of length in the length AB is the limit to which $\Sigma \chi$ tends when the number n is increased indefinitely, and the lengths of all the chords are diminished indefinitely. In case x always increases, as a point travels along the curve from A to B, we have the result

$$s = \int_a^b \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} dx,$$

s being the number of units of length in the length of AB, and a, b the x -coordinates of A, B. The result may be expressed by saying that the length of a curve is the limit of the length of the perimeter of an inscribed polygon. The limit is arrived at by making the lengths of all the sides of the polygon tend to zero.

In the case of a surface of revolution, generated by the revolution of an arc AB about an axis, we divide AB as before. The chord joining two of the points such as P_{r-1} and P_r generates a belt of a cone. Let the area of this belt be σ_r units of area. Then the area of the surface of revolution is the limit to which $\Sigma \sigma_r$ tends. If the curve revolves about the axis of x , σ_r is approximately equal to $2\pi y_r \chi_r$. The exact value is expressible as $2\pi y_r' \chi_r$ where y_r' tends to y_r as a limit when χ_r tends to zero. This leads to the result obtained in §§ 148, 149. The result may be otherwise expressed by saying that the area of the surface is the limit of the area of a different surface, viz.: one obtained by inscribing a polygon in the generating curve and making this polygon revolve about the axis. The limit is arrived at in the same way as in the case of an arc.

In the next Chapter we shall consider further examples of limits of sums evaluated as definite integrals.

CHAPTER XI

SOME APPLICATIONS OF DEFINITE INTEGRALS TO MECHANICS

157. WE may use the Integral Calculus to determine the position of the centre of gravity of a body. For this purpose we begin by recalling the meaning attached in Statics to certain terms.

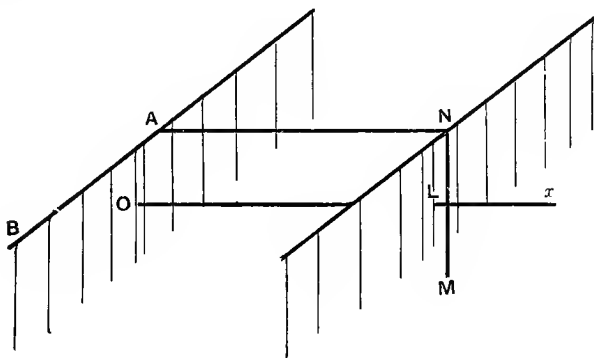


Fig. 70.

We think of a force of magnitude W lbs.¹ acting downwards in a vertical straight line NM , and we think of a horizontal

¹ In regard to the units of force &c., see Appendix VII.

straight line AB which does not intersect NM. In Fig. 70 AB is to be thought of as being at right angles to the plane of the paper. A horizontal plane passing through AB would meet NM in a point, say N, and a perpendicular from N to AB would meet AB in a point, say A. If the length of AN is x feet, the product xW is the measure in lb.-ft. units of the "moment" of the force about the straight line AB.

We could make pass through AB and NM two vertical planes parallel to each other. The x which occurs in the product xW is the number of feet in the distance between these two planes.

It is usual to give a sign to the moment so as to indicate the sense in which the force tends to turn a body about the straight line AB. We could take an origin O on the vertical plane through AB, and an axis of x at right angles to this plane. The moment with its proper sign is still measured in lb.-ft. units by the product xW , if x means the x -coordinate of the point L in which the vertical plane, passing through NM and parallel to AB, is cut by the axis of x .

158. The force of the Earth's gravity acting on a body does not act at one point of the body more than at another; but, if the body is rigid, it can be supported by a single force, without undergoing any change of size or shape, provided this force is directed vertically upwards and acts in the proper line. However the body may be turned about, there is one point (fixed with respect to the body) through which the line in question always passes. This point is the "centre of gravity" of the body. For many purposes we may regard the force of the Earth's gravity acting on a body as a single force acting downwards in the vertical straight line which passes through the centre of gravity. This force is often called the "weight" of the body.

159. We may think of a body as consisting of several parts. Each part is a body having a certain weight and a certain centre

of gravity. The rule by which the centre of gravity of a body is determined is this:—The moment (about any horizontal axis) of the weight of the body, acting at the centre of gravity of the body, is equal to the sum of the moments (about the same axis) of the weights of the parts, acting at the centres of gravity of the parts.

Let the x -coordinates of the centres of gravity of the parts be x_1', x_2', \dots , and their weights W_1, W_2, \dots lbs. Let the x -coordinate of the centre of gravity of the body be x_G . Then we have the equation $x_G (W_1 + W_2 + \dots) = x_1' W_1 + x_2' W_2 + \dots$,

or, as it may be written, $x_G \Sigma W = \Sigma x' W$.

The coefficient ΣW in the left-hand member of this equation is the numerical measure in lbs. of the weight of the body.

In applying this formula to find the centre of gravity of a specified body we shall think of the body as "homogeneous" or "uniform," meaning that the weight of any volume is proportional to the volume. We evaluate such a sum as $\Sigma x' W$ by passing to a limit, the volumes of all the parts being diminished indefinitely. The limit of such a sum is expressed by a definite integral.

160. As an example we find the centre of gravity of a hemisphere. Let the hemisphere be placed with its plane base vertical, and let the radius be r feet. The section of the surface by a plane parallel to the base, and at a distance x feet from it, is a circle, and the area of the circle is $\pi (r^2 - x^2)$ square feet (Fig. 71). The volume of the slice between two such planes specified by x and $x + \Delta x$ is intermediate between $\pi (r^2 - x^2) \Delta x$

and $\pi \{r^2 - (x + \Delta x)^2\} \Delta x$

cubic feet. The distance, x' feet, of the centre of gravity of the

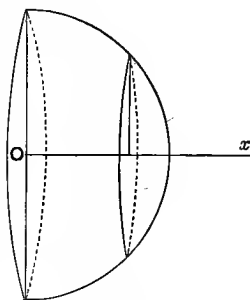


Fig. 71.

slice from the base is intermediate between x and $x + \Delta x$ feet. Let the weight of a cubic foot of the substance be σ lbs. As all the numbers Δx tend to zero, the sum expressed by $\Sigma x'W$ tends to a limit, which is

$$\int_0^r \sigma \pi (r^2 - x^2) x dx,$$

and ΣW is $\frac{2}{3} \pi r^3 \sigma$. Hence we have

$$x_G \frac{2}{3} \pi r^3 \sigma = \sigma \pi \left(r^2 \frac{r^2}{2} - \frac{r^4}{4} \right) = \frac{1}{4} \sigma \pi r^4,$$

and

$$x_G = \frac{3}{8} r.$$

The distance of the centre of gravity from the base is $\frac{3}{8}$ of the radius. The centre of gravity obviously lies on the straight line drawn through the centre of the base at right angles to the base.

161. In the case of a body in general, if a, b are the least and greatest values of x that occur, if the area of the section of the body by a plane at a distance x feet from the origin is Z square feet, and the volume of the body V cubic feet, the same reasoning leads to the result

$$x_G V = \int_a^b Z x dx.$$

162. For another example, let the body be a pyramid on a triangular base.

Let the area of the base be B square feet, and the height p feet. As in § 80 the area of the section specified by x is $B \frac{x^2}{p^2}$ square feet, and the volume of the pyramid is $\frac{1}{3} Bp$ cubic feet. We have

$$x_G \frac{1}{3} Bp = \int_0^p B \frac{x^2}{p^2} x dx = \frac{1}{4} Bp^2,$$

and

$$x_G = \frac{3}{4} p.$$

Hence the distance of the centre of gravity from the base is $\frac{1}{4}$ of the distance of the opposite vertex from the base. This result holds whichever face we take as base. It is equivalent to the result that the centre of gravity of the pyramid coincides with that of four equal weights placed at its corners.

163. The point which we determine as the centre of gravity of a uniform solid body is a sort of centre of a solid figure, the figure of the body. It is called the "centroid" of the figure. In like manner a plane figure has a centroid, which coincides with the centre of gravity of a very thin slab or lamina, having the shape of the figure, and having a uniform small thickness as well as a uniform density. Let the area of the figure be S units of area, and let the least and greatest values of x that occur in the figure be x_0 and x_1 . Let any straight line be drawn across the figure parallel to the axis of y , and let Y units of length be the length intercepted by the figure upon this straight line, so that Y is a function of x (Fig. 72). Then the x of the centroid is given by the equation

$$x_G S = \int_{x_0}^{x_1} xY dx.$$

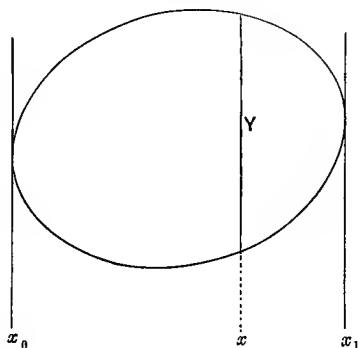


Fig. 72.

In like manner, if y_0 and y_1 are the least and greatest values of y that occur in the figure, and X units of length is the length cut out by the figure on a straight line parallel to the axis of x , so that X is a function of y (Fig. 73), the y of the centroid is given by the equation

$$y_G S = \int_{y_0}^{y_1} X y dy.$$

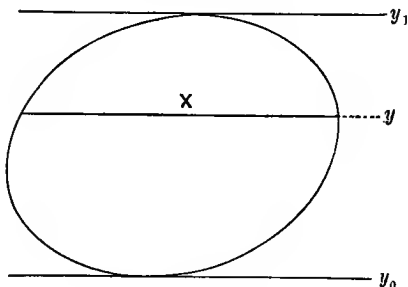


Fig. 73.

EXAMPLES

1. Use the method of the Integral Calculus to prove that the distance of the centroid of a triangle from any side of the triangle is $\frac{1}{3}$ of the distance of the opposite vertex from that side.

2. In the case of a semicircle, let the axis of y be the bounding diameter and the axis of x the straight line drawn at right angles to it through the centre of the circle, r units of length the radius. In the notation of § 163 $y_G = 0$ by symmetry, and $S = \frac{1}{2} \pi r^2$, $Y = 2\sqrt{(r^2 - x^2)}$, $x_0 = 0$, $x_1 = r$. Prove that $x_G = \frac{4}{3\pi} r$, or the distance of the centroid from the centre is 0.4244 of the radius approximately.

3. In the case of a segment of a circle, let the bounding chord of the segment subtend at the centre an angle 2α radians, let the origin be at the centre of the circle, and let the axis of x bisect the bounding chord. Prove that $S = r^2 (\alpha - \sin \alpha \cos \alpha)$, $y_G = 0$, $x_G = \frac{2}{3} r \frac{\sin^3 \alpha}{\alpha - \sin \alpha \cos \alpha}$.

4. In the case of a sector of a circle, the two bounding semi-diameters containing an angle 2α radians, let the origin be at the centre of the circle, and let the axis of x bisect the angle of the sector. Prove that $y_G = 0$ and

$$x_G = \frac{2}{3} r \frac{\sin \alpha}{\alpha}.$$

5. We may define the centroid (x_G, y_G) of an arc by the formulae

$$x_G \cdot l = \int_{s_0}^{s_1} x ds, \quad y_G \cdot l = \int_{s_0}^{s_1} y ds, \quad \text{where the length of the arc is } l \text{ units of}$$

length, x, y are the coordinates of a point of the arc distant s units of length along the curve from some fixed point of the curve, and s_0, s_1 are the extreme values of s . The centroid of the arc coincides with the centre of gravity of a piece of uniform wire bent into the shape of the arc. Prove that in the case of a circular arc, subtending an angle 2α radians at the centre of the circle, the centroid is on the straight line drawn through the centre to bisect this angle, and is at a distance from the centre equal to $\frac{\sin \alpha}{\alpha}$ of the radius.

CENTRES OF PRESSURE

164. Centroids are important in Hydrostatics as well as in Statics. The pressure of a fluid at a point is measured as so many lbs. per square foot. The pressure of water (at rest) at a point distant y feet below the surface of the water is wy lbs. per square foot, where w denotes the weight of a cubic foot of water. A unit of pressure is one lb. per square foot.

Consider the force with which water presses against a limited portion of a vertical plane with which it is in contact. This force is called the "resultant thrust" of the water against the portion of the plane. We shall refer to the portion of the plane as the "figure." Let the area of the figure be S units of area. The plane cuts the water surface in a horizontal line. On this line take an origin O , and draw the axis of y vertically downwards. The figure will be contained between two horizontal lines at depths a and b feet below the water line (Fig. 74). Any intermediate horizontal line will cut the boundary of the figure, and there will be cut out upon it a certain length, say X feet. The pressure at any point between the lines specified by y and

$y + \Delta y$ will be intermediate between wy and $w(y + \Delta y)$ lbs. per square foot, and we may take it to be wy' lbs. per square foot, where y' is some number between y and $y + \Delta y$. The area between the two lines is $X'\Delta y$ square feet, where X' lies between the greatest and least values of X that belong to lines between those specified by y and $y + \Delta y$. Hence the resultant thrust on this part of the figure is $wy'X'\Delta y$ lbs. The resultant thrust on the whole figure is, in lbs.,

$$w \sum_a^b y'X'\Delta y.$$

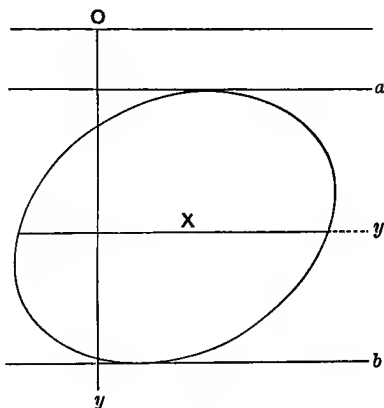


Fig. 74.

As all the numbers Δy tend to zero this passes over into

$$w \int_a^b yX dy.$$

But if y_G is the y of the centroid of the figure we have, as in § 163,

$$y_G S = \int_a^b yX dy.$$

The resultant thrust is therefore $wy_G S$ lbs. Now the pressure at the centroid is wy_G lbs. per square foot. Hence the resultant thrust is the same as it would be if the pressure were the same at all points of the figure and equal to the pressure at the centroid.

165. The thrust of the water against the figure is statically equivalent to a single force. We have just found the magnitude of this force. Its line of action cuts the plane of the figure in a point, called the "centre of pressure." The depth of the centre of pressure below the water line is determined by the rule:—If the figure is regarded as made up of parts, the moment (about the water line) of the resultant thrust on the whole figure, acting at the centre of pressure of the figure, is equal to the sum of the moments (about the water line) of the resultant thrusts on the parts, acting at the centres of pressure of the parts.

As a convenient part of the figure we take the strip between two horizontal lines at depths y and $y + \Delta y$ feet below the water line. We know that its area can be expressed as $X' \Delta y$ square feet, and the resultant thrust against it as $wy_G' X' \Delta y$ lbs., if its centroid is at a depth y_G' feet. Let the depths of the centre of pressure of the whole figure and of the strip be y_P and y_P' feet. We have the equation

$$y_P w y_G S = w \sum y_P' y_G' X' \Delta y.$$

Since the centre of pressure of the strip must, from the nature of the case, be a point on the strip, the limit to which y_P' tends as Δy tends to zero is y . Hence the equation becomes

$$y_P y_G S = \int_a^b y^2 X dy.$$

EXAMPLES

1. In the case of a triangle with one side in the water surface, let the length of this side, BC in Fig. 75, be a feet, and let the distance of the opposite vertex A from BC be p feet. Since the triangles APQ , ABC are similar, we have

$$\frac{x}{a} = \frac{p-y}{p}.$$

Also $y_G = \frac{1}{3}p$ and $S = \frac{1}{2}ap$.

Hence
$$y_P \frac{1}{6}ap^2 = \int_0^p y^2 \frac{a}{p}(p-y) dy$$

$$= \frac{a}{p} \left(\frac{1}{3}p^4 - \frac{1}{4}p^4 \right) = \frac{1}{12}ap^3,$$

and therefore
$$y_P = \frac{1}{2}p,$$

or the depth of the centre of pressure is $\frac{1}{2}$ the depth of the lowest point.

2. If one side of a rectangle is in the water surface the depth of the centre of pressure of the rectangle is $\frac{2}{3}$ that of the opposite side.

3. If the centre of a circle is in the water surface, the depth of the centre of pressure of the immersed semicircle is $\frac{3}{16}\pi$ of the radius, or 0.589 of the radius approximately. [For the integration cf. § 135 (ii).]

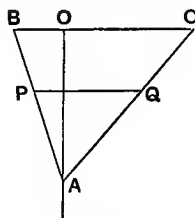


Fig. 75.

MOMENTS OF INERTIA

166. The motion of a very small body may be specified by the motion of one point of it. If w lbs. is the weight of the small body, and v feet per second its velocity, its kinetic energy is $\frac{1}{2} \frac{w}{g} v^2$ foot-pounds, where g stands for the number 32.2. If the body is moving round a circle, so that the line drawn from the

centre of the circle to the position of the body turns with an angular velocity ω radians per second, and the radius of the circle is r feet, $v = r\omega$, and the kinetic energy is $\frac{1}{2} \frac{w}{g} r^2 \omega^2$ foot-pounds.

We may think of a moving body in general as made up of small parts, the motion of each of which can be specified by the motion of one point in it. The kinetic energy of the body is the sum of the kinetic energies of the small parts.

If the body is rotating about an axis we may take the points, by the motions of which the motions of the parts are specified, to describe circles about this axis, and the angular velocity will be the same for all of them. Let this angular velocity be ω radians per second. Let the weights of the parts be w_1, w_2, \dots lbs. and the radii of the circles r_1, r_2, \dots feet. The kinetic energy of the body is $\frac{1}{2} \omega^2 \left(\frac{w_1}{g} r_1^2 + \frac{w_2}{g} r_2^2 + \dots \right)$ foot-pounds. The sum $\frac{w_1}{g} r_1^2 + \frac{w_2}{g} r_2^2 + \dots$, or, as it may be written, $\Sigma \frac{w}{g} r^2$, is the measure in lb.-ft. units of a certain quantity called the "moment of inertia" of the body about the axis.

We shall suppose the body to be homogeneous. In finding the moment of inertia we may first suppose that the body is divided into thin sheets by means of a series of cylinders, having the axis about which the body turns as a common axis. All the parts between two cylinders, whose radii are r and $r + \Delta r$ feet, will have velocities intermediate between $r\omega$ and $(r + \Delta r)\omega$ feet per second. If r' is some suitable number between r and $r + \Delta r$, and w' lbs. is the weight of all these parts, their contribution to $\Sigma \frac{w}{g} r^2$ is $\frac{w'}{g} r'^2$.

If a cubic foot of the body weighs σ lbs. this is the same as $\frac{\sigma}{g} r'^2 \Delta v$, where Δv cubic feet is the volume contained between

the two cylinders. Then the expression for the moment of inertia becomes $\frac{\sigma}{g} \sum r'^2 \Delta v$.

167. Let the body be a circular disc, of uniform thickness h feet and radius a feet, rotating about an axis passing through its centre at right angles to the planes of its faces. Then $\Delta v = h 2\pi \left(r + \frac{1}{2} \Delta r\right) \Delta r$. (Cf. § 58 (b).) Now let all the numbers Δr tend to zero, then $\sum r'^2 \Delta v$ tends to a limit, which is

$$\int_0^a 2\pi h r \cdot r^2 \cdot dr,$$

or $\frac{1}{2} \pi h a^4$. The moment of inertia of the body is, in lb.-ft. units, $\frac{1}{2} \pi \frac{\sigma}{g} h a^4$. Since $\sigma \pi a^2 h$ lbs. is the weight of the body, say W lbs., we find that the moment of inertia is in lb.-ft. units $\frac{1}{2} \frac{W}{g} a^2$.

168. The moment of inertia of a body about an axis can generally be expressed in the form $\frac{W}{g} k^2$ lb.-ft. units, where W lbs. is the weight of the body, g is the number 32.2, and k is a number depending on the shape and size of the body. The length k feet is then called the "radius of gyration" of the body about the axis. The kinetic energy of the body is the same as if all the matter in it were condensed uniformly upon the circumference of a circle, of radius k feet, with its centre on the axis about which the body rotates, and its plane at right angles to this axis. If V cubic feet is the volume of the body, Δv cubic feet the volume of the part of it that is distant less than $r + \Delta r$ feet from the axis, and more than r feet from the axis, then $V k^2$ is equal to the limit of the sum $\sum r^2 \Delta v$, the limit being arrived at by making all the numbers Δr tend to zero.

The result obtained in § 167 is that the radius of gyration of a circular disc (radius a feet) about an axis passing through its centre at right angles to the planes of its faces is $\frac{1}{2} a \sqrt{2}$ feet.

169. In like manner we may define the radius of gyration (k units of length) of a plane figure about an axis in its plane. With the notation of § 163, the radius of gyration about the axis of y is given by the equation

$$k^2 S = \int_{x_0}^{x_1} x^2 Y dx.$$

170. As an example, let the figure be a rectangle, let the lengths of its sides be $2a$ and $2b$ units of length, and let the axis of y be a straight line drawn through the centroid of the rectangle parallel to the sides specified by $2b$ (Fig. 76). Then we have $S = 4ab$, $Y = 2b$, $x_0 = -a$, $x_1 = a$, and

$$4k^2 ab = \int_{-a}^a 2bx^2 dx = \frac{4}{3} ba^3, \text{ or } k^2 = \frac{1}{3} a^2.$$

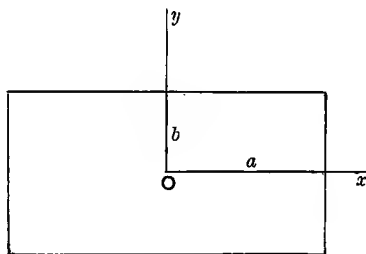


Fig. 76.

As a second example let the figure be a circle, of radius a units of length, and let the axis of y pass through the centre. Then we have

$$S = \pi a^2, \quad Y = 2 \sqrt{(a^2 - x^2)}, \quad x_0 = -a, \quad x_1 = a,$$

$$\text{and } \pi a^2 k^2 = \int_{-a}^a 2x^2 \sqrt{(a^2 - x^2)} dx = \frac{1}{4} \pi a^4 \text{ (§ 135 (ii))}, \text{ or } k^2 = \frac{1}{4} a^2.$$

The radius of gyration of a plane figure about an axis in its plane is important in connexion with the theory of resistance of a beam to bending. We have found also in § 165 that the depth of the centre of pressure of a figure is determined by finding the depth of the centroid and the radius of gyration about the water-line.

TABLE OF STANDARD FORMS OF INTEGRALS.

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APPENDIX

I. THE GRAPH OF A RATIONAL INTEGRAL FUNCTION OF THE FIRST DEGREE

It is to be shown that all the points whose coordinates satisfy the equation $y = mx + b$ lie on a straight line. Let A, B, C be three of the points, named in such an order that $x_C > x_B > x_A$. Since $y = mx + b$, we have

$$\frac{y_C - y_B}{x_C - x_B} = \frac{y_B - y_A}{x_B - x_A} = \frac{y_C - y_A}{x_C - x_A} = m,$$

and therefore, if $y_C > y_B$, $y_B > y_A$, and, if $y_C < y_B$, $y_B < y_A$.

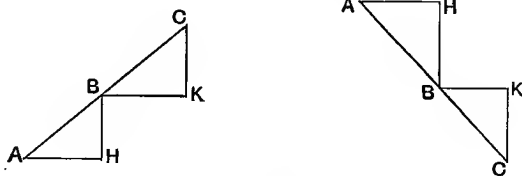


Fig. 77.

The two cases are illustrated in Fig. 77. On AB and BC place right-angled triangles AHB, BKC with their sides AH, BK, parallel to the axis of x and their sides HB, KC parallel to the axis of y . The equation

$$\frac{y_C - y_B}{x_C - x_B} = \frac{y_B - y_A}{x_B - x_A}$$

gives the proportion $CK : BK = BH : AH$, also the angles CKB , BHA are right angles, and therefore the triangles CKB , BHA are similar, and the angle CBK is equal to the angle BAH . Hence A , B , C are in one straight line.

Conversely, it is to be shown that the coordinates of all the points on a straight line (not parallel to either coordinate axis) satisfy an equation of the form $y = mx + b$. Let P be the point where the straight line meets the axis of y , and let its coordinates be $0, b$; this fixes b . Let Q be the point where the straight line meets the axis of x , and let its coordinates be $-\frac{b}{m}, 0$; this fixes
 m . Let $R(x, y)$ be any other point on the straight line. The points P, Q, R , in some order, are situated like the points A, B, C of Fig. 77. Since these points are in a straight line, we have the proportion

$$CK : BK = BH : AH,$$

which gives the equation

$$\frac{y_C - y_B}{x_C - x_B} = \frac{y_B - y_A}{x_B - x_A},$$

and from this equation we deduce the equation

$$\frac{y_B - y_A}{x_B - x_A} = \frac{y_C - y_A}{x_C - x_A}.$$

Now the three quotients, in some order, are the same as

$$\frac{y}{x + \frac{b}{m}}, \quad m, \quad \frac{y - b}{x},$$

and the condition of equality of any two of these is $y = mx + b$.

II. LIMITS

In general the proof that some number L is the limit to which a function $F(h)$ tends as h tends to zero is to be found by setting up some positive number ϵ , as small as may be wished, and showing

that the absolute value of the difference $F(h) - L$ can be made less than ϵ by bringing h near enough to zero. If this can be done for every ϵ , however small, the limit is L .

In the example worked out in § 20 we had to make h less than $\frac{\epsilon}{4x}$, and then we showed that the difference was less than ϵ if h was any number between 0 and $\frac{\epsilon}{4x}$. In general, by making h small enough is meant bringing h to lie between zero and some positive number k , if h is positive, or to lie between zero and $-k$, if h is negative. In the example k was $\frac{\epsilon}{4x}$.

We can put the general definition in the following form:—Let h be a variable which tends to zero, $F(h)$ a function of h , L the limit to which it tends as h tends to zero. Any positive number ϵ , as small as may be wished, is chosen. Another positive number k is then found so that, if h is any number whatever between 0 and k , or between 0 and $-k$, the absolute value of $F(h) - L$ is less than ϵ . When this can be done the right number L to be the limit is distinguished from all other numbers.

We may show how the definition includes the results that $\frac{dy}{dx} = 0$ when $y = a$, and $\frac{dy}{dx} = 1$ when $y = x$, which were noted in § 17.

First let $y = a$, a number independent of x . When x is changed to $x + h$, y is not changed. Writing $f(x)$ for y we have

$$f(x) = a, \quad f(x + h) = a,$$

and therefore

$$\frac{f(x + h) - f(x)}{h} = 0$$

for all values of h other than 0. In this case $L = 0$. In fact we may take any positive number we please for k , and it is true that

$$0 - \frac{f(x + h) - f(x)}{h} < \epsilon$$

if h has any value between 0 and k or between 0 and $-k$, for $0 - \frac{f(x + h) - f(x)}{h}$ is precisely zero.

Next let $y = x$. When x is changed to $x + h$, y is changed to $y + h$, and, writing $f(x)$ for y , we have

$$\frac{f(x+h) - f(x)}{h} = 1$$

for all values of h other than 0. In this case $L=1$. We may take any positive number we please for ϵ , and it is true that

$$1 - \frac{f(x+h) - f(x)}{h} < \epsilon$$

if h has any value between 0 and k or between 0 and $-k$.

We may define in a similar way the limit to which a function $F(x)$ tends as x tends to a particular value a . The difference $x - a$ may be positive or negative, but if, as $x - a$ tends to zero, $F(x)$ tends to a limit L , the function $F(x)$ tends to the limit L as x tends to a . The definition may be written more at length as follows:—Let ϵ be any positive number, as small as may be wished. If a positive number k can be found so that, for all values of x which lie between a and $a + k$, or between a and $a - k$, the absolute value of the difference $F(x) - L$ is less than ϵ , $F(x)$ tends to L as a limit when x tends to a .

The statement in § 13, that all the functions which we consider possess graphs, implies a certain limitation to which all the functions are subject. If $y = f(x)$, and $f(x)$ has a graph, then when x is changed to $x + \Delta x$ and y to $y + \Delta y$, it is implied that Δy tends to zero as Δx tends to zero. According to the above definition this is the same thing as saying that, as x tends to any value a , $f(x)$ tends to a limit, which is $f(a)$. A function which is subject to this limitation is said to be "continuous." The only kind of discontinuity which is met with in elementary work is the kind which occurs if there is a value of x for which the function $f(x)$ cannot be calculated, because the formation of the expression for $f(x)$ would require division by zero. For example, the function $\frac{1}{x}$ is not continuous when $x = 0$. The possibility of such values implies a restriction upon the generality of some theorems. The theorem of § 56 can be proved to hold for any function $f(x)$ if the derived function $f'(x)$ is continuous

for all values of x between a and b inclusive. The theorem (§ 154) that the definite integral $\int_a^b F'(x) dx$, considered as the limit of a sum such as $\sum_a^b F'(x) \Delta x$, is equal to $F(b) - F(a)$, can be proved to hold for any function $F(x)$, if the derived function $F'(x)$ is continuous for all values of x between a and b . In elementary work these restrictions are always understood.

We prove here a series of four theorems concerning limits. All of them are nearly obvious, but it is satisfactory to prove them from the definition.

(1) If, as h tends to zero, $F(h)$ tends to L as a limit, then $aF(h)$ tends to aL as a limit, a being any constant.

Let η be any positive number, as small as we please. We know that a positive number k can be found so that, when h lies between 0 and k , or between 0 and $-k$, the absolute value of $F(h) - L$ is less than η . Let $\epsilon = a\eta$ or $-a\eta$ according as a is positive or negative. Then ϵ can be any positive number, as small as may be wished. The absolute value of $aF(h) - aL$ is less than ϵ . Therefore aL is the limit of $aF(h)$.

A special case of this theorem was used in proving the first Rule of differentiation (§ 21).

(2) If, as h tends to zero, $F_1(h)$ and $F_2(h)$ tend to L_1 and L_2 as limits, then $F_1(h) + F_2(h)$ tends to $L_1 + L_2$ as a limit.

Let η be as before. We know that a positive number k_1 can be found so that, when h lies between 0 and k_1 , or between 0 and $-k_1$, the absolute value of $F_1(h) - L_1$ is less than η . We know also that a positive number k_2 can be found so that, when h lies between 0 and k_2 , or between 0 and $-k_2$, the absolute value of $F_2(h) - L_2$ is less than η . Let k be a positive number less than either k_1 or k_2 , and let $2\eta = \epsilon$. Then ϵ can be any positive number, as small as may be wished, and we know that the absolute value of

$$F_1(h) - L_1 + F_2(h) - L_2 \text{ or of } \{F_1(h) + F_2(h)\} - (L_1 + L_2)$$

is less than ϵ , when h lies between 0 and k or between 0 and $-k$. Hence $F_1(h) + F_2(h)$ tends to a limit, and $L_1 + L_2$ is that limit.

A special case of this theorem was used in proving the second Rule of differentiation (§ 21).

When there are three terms $F_1(h)$, $F_2(h)$, $F_3(h)$ and their limits are L_1 , L_2 , L_3 , we have

$$\text{limit of } \{F_1(h) + F_2(h)\} = (L_1 + L_2).$$

$$\text{Now } F_1(h) + F_2(h) + F_3(h) = \{F_1(h) + F_2(h)\} + F_3(h).$$

The sum of three terms is expressed as the sum of two which have limits $(L_1 + L_2)$ and L_3 . Hence

$$\text{limit of } F_1(h) + F_2(h) + F_3(h) = L_1 + L_2 + L_3.$$

In the same way the theorem may be proved for a sum of n terms if n is any whole number.

(3) If, as h tends to zero, $F_1(h)$ and $F_2(h)$ tend to L_1 and L_2 as limits, then $F_1(h) \cdot F_2(h)$ tends to $L_1 L_2$ as a limit.

$$\text{Let } F_1(h) = L_1 + X, \quad F_2(h) = L_2 + Y.$$

We know that X and Y tend to zero as a limit. Now

$$\begin{aligned} F_1(h) \cdot F_2(h) &= (L_1 + X)(L_2 + Y) \\ &= L_1 L_2 + L_1 Y + L_2 X + XY. \end{aligned}$$

When h is chosen as in Theorem (2), the absolute value of XY is less than η^2 , and this is less than η if η is less than 1. Hence the product XY tends to zero as a limit. By Theorem (1) $L_1 Y$ and $L_2 X$ tend to zero as a limit. Therefore by Theorem (2)

$$L_1 Y + L_2 X + XY$$

tends to zero as a limit. Hence $F_1(h) \cdot F_2(h)$ tends to $L_1 L_2$ as a limit.

A special case of this theorem was used in proving the Rule for differentiating a product.

Just as in Theorem (2) the result may be extended to the product of n factors if n is any whole number.

(4) If, as h tends to zero, $F(h)$ tends to L as a limit, and if L is not zero, $\frac{1}{F(h)}$ tends to $\frac{1}{L}$ as a limit.

Let η be any positive number, as small as may be wished. We know that a number k can be found so that, when h lies between 0 and k , or between 0 and $-k$, the absolute value of $F(h) - L$ is less than η . Now

$$\frac{1}{L} - \frac{1}{F(h)} = \frac{F(h) - L}{LF(h)}.$$

To fix ideas let L be positive, and let $F(h) = L + X$.

When h is as above, the absolute value of X is less than η , and the absolute value of $F(h)$ is greater than $L - \eta$. Hence the absolute value of $\frac{1}{L} - \frac{1}{F(h)}$ is less than $\frac{\eta}{L(L - \eta)}$. If we put ϵ for this, ϵ can be any positive number, as small as may be wished. We have therefore proved that, when ϵ is chosen, a positive number k can be found so that, when h lies between 0 and k , or between 0 and $-k$, the absolute value of $\frac{1}{F(h)} - \frac{1}{L}$ is less than ϵ .

A special case of this theorem was used in § 25 in proving the Rule $\frac{dy}{dx} \frac{dx}{dy} = 1$.

In the investigation of limits it is often convenient to introduce a set of numbers $u_1, u_2, \dots u_n, \dots$ with some rule by which the n th number of the set, u_n , can be expressed in terms of the integer n . Such a set is called a "sequence." The sequence may have a limit L . Let any positive number ϵ , however small, be chosen, and let a number N be found so that, when n is any integer greater than N , the absolute value of $u_n - L$ is less than ϵ . Then L is the limit of the sequence. For example, if $u_n = \left(1 + \frac{1}{n}\right)^n$, L is e , as will be proved below. Let ϵ be as above; and suppose that we can find an integer N so that, if m and

n are any integers greater than N , the absolute value of $u_m - u_n$ is less than ϵ , then we know that the sequence has a limit.

Many theorems concerning limits are proved by using two sequences $x_1, x_2, \dots, x_n, \dots$ and $y_1, y_2, \dots, y_n, \dots$. If we can show (i) that as n increases x_n increases and y_n diminishes, (ii) that every one of the numbers y is greater than any one whatever of the numbers x , (iii) that the difference $y_n - x_n$ can be made as small as we please by increasing n sufficiently, we can infer that both sequences have the same limit. When it is said that, as n increases, x_n increases, it is not meant that necessarily $x_2 > x_1$, $x_3 > x_2$, and so on. What is meant is that, whenever a change occurs as n increases, the change is an increase. Similarly for the y 's. Now let ϵ be any positive number, as small as may be wished, let κ be an integer such that $y_\kappa - x_\kappa < \epsilon$, and let m and n be any integers greater than κ . For clearness take m to be greater than n . Then $y_m \leq y_n \leq y_\kappa$ and $x_m \geq x_n \geq x_\kappa$, and therefore $x_m - x_n < \epsilon$ and $y_n - y_m < \epsilon$. This proves the statement.

We may illustrate this method geometrically by taking the x 's and the y 's to be the numerical measures of distances of points in a straight line from a fixed point in the line. To fix ideas, we are supposing all the numbers x and y to be positive. As the suffix increases the X point always

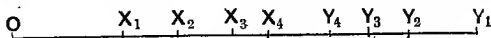


Fig. 78.

moves to the right, and the Y point to the left (Fig. 78). When n is large enough the distance between an X point and the corresponding Y point is very short. All the subsequent points of both sets are crowded into this very short length, and yet every Y point is to the right of all the X points.

As an example of the application of this method let $u_1, u_2, \dots, u_n, \dots$ be a sequence of positive terms, let u_n increase as n increases, and let it be known that no term is so great as a certain number A . The sequence has a limit which is either the number A or a number less than A . We form two new sequences x_1, x_2, \dots and y_1, y_2, \dots . Let x_1 be u_1 and y_1 be A .

Consider the number half way between x_1 and y_1 , it is $u_1 + \frac{1}{2}(A - u_1)$, denote it by B_1 . Either some number of the sequence u is greater than B_1 or else no number of the sequence u is greater than B_1 . In the first case let x_2 be B_1 and let y_2 be the same as y_1 ; in the second let x_2 be the same as x_1 and let y_2 be B_1 . Choose in the same way a number B_2 half way between x_2 and y_2 and form with it the numbers x_3, y_3 . At every step we halve the difference $y_1 - x_1$, and after $n - 1$ steps we have $y_n - x_n = \frac{y_1 - x_1}{2^{n-1}}$, which can be made as small as we please by increasing n sufficiently. Now after $n - 1$ steps there are numbers of the sequence u which are greater than x_n but no numbers of the sequence u which are greater than y_n . The numbers of the sequence u which lie between x_n and y_n are those whose suffixes exceed some particular integer N . It appears therefore that by taking N great enough and l and m to be integers greater than N we can make the absolute value of the difference $u_l - u_m$ as small as we please. Hence the sequence u has a limit.

We may illustrate this process geometrically. Finding B_1 is bisecting the length X_1Y_1 . Finding B_2 is bisecting the length X_1B_1 or B_1Y_1 according as there are not or are to the right of B_1 points specified by some u . To

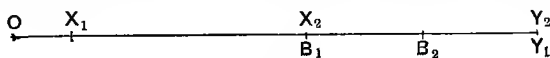


Fig. 79.

fix ideas suppose that B_1Y_1 is bisected. Then the next step is to bisect B_1B_2 or B_2Y_1 according as there are not or are to the right of B_2 points specified by some u . We very soon find all the values of u_n for high values of n confined between two points which are very close together, and we can bring them as close together as we please, thus crowding all the values of u_n for high values of n close to one point—the limit to the left of which all the points corresponding to values of u_n lie.

In like manner if u_n always diminishes as n increases, but remains always greater (algebraically) than some fixed number A , the sequence has a limit, which is either A or a number algebraically greater than A .

III. INDICES AND LOGARITHMS

The definition of $\log_{10} x$ by the equations $10^y = x$, $y = \log_{10} x$ is in some ways incomplete. The question might be asked: How is it known that to a given number x there corresponds a number y which makes 10^y equal to x ? If x happens to be 1 or 10, or $\frac{1}{10}$, or 100, or $\frac{1}{100}$, or any positive or negative integral power of 10, y is known. If x happens to be the square or cube root of 10, or any other root say the q th root, or if it is the p th power of the q th root, y is known. But some numbers are not powers of roots of 10. If for instance x were 2, no fraction $\frac{p}{q}$ could be y . If $10^{\frac{p}{q}}$ were 2, 10^p would be 2^q , and 5^p would be 2^{q-p} , an odd number would be equal to an even number, which cannot happen. Then it might be said that, for that matter, the square root of 10 and many other numbers are not fractions like $\frac{p}{q}$, and that when we speak of 10^y as being equal to 2 we must mean y to be a number of this sort. But, if this is said, the question may be asked: What does 10^y mean if y is not an integer or a fraction? 10^y was defined by the law of indices for integers and fractions, but it has not been defined for any other numbers.

The question can be answered by taking proper account of the true meaning of such numbers as $10^{\frac{1}{2}}$, "irrational" numbers, as they are called, to distinguish them from the ordinary "rational" numbers. Integers, and fractions of which the numerators and denominators are integers, together constitute the class of numbers called rational numbers. Now when we find by the ordinary arithmetical process the square root of 10 correctly to a number of places of decimals we really show successively that the numbers 4, 3·2, 3·17, 3·163, 3·1623 and so on have their squares greater than 10, but that the numbers 3, 3·1, 3·16, 3·162,

3.1622 and so on have their squares less than 10. That is to say, we gradually sort the rational numbers into two sets: a set of which every one has its square greater than 10, and a set of which every one has its square less than 10; and the square root of 10 means a number which is less than every number of the first set and greater than every number of the second set. An irrational number like $10^{\frac{1}{2}}$ may properly be said to be "known" whenever we have a process for determining, in regard to any rational number, whether that rational number is greater or less than the irrational number in question.

If y is a rational number of the form $\frac{2m+1}{2^n}$, where m and n are integers, 10^y is a known irrational number.

If y (supposed positive) is any known irrational number, or any rational number that is neither an integer nor of the form $\frac{2m+1}{2^n}$, and if a and b are any two rational numbers of this form, such that $a > y$ but $b < y$, we define 10^y as a number which is greater than 10^b but less than 10^a . We can then say, in regard to any rational number γ , whether 10^y is greater than, equal to, or less than γ . Take any number a from the a set and any number b from the b set. Then, if $\gamma > 10^a$, $10^y < \gamma$; and, if $\gamma < 10^b$, $10^y > \gamma$. If however $10^a > \gamma > 10^b$, we choose a smaller number a from the a set and a larger number b from the b set. It may happen that every number of the set 10^a is greater than γ and every number of the set 10^b is less than γ . In this case 10^y is equal to γ . But, if this is not the case, either (i) γ is greater than some number of the set 10^a , or (ii) γ is less than some number of the set 10^b . We can therefore sort the rational numbers such as γ into two sets: those which are greater than 10^y and those which are less. There cannot be two rational numbers γ_1 and γ_2 which are in neither set, because we can take a and b so near together that $10^a - 10^b$ is less than the absolute value of $\gamma_1 - \gamma_2$. If there is one rational number which is in neither set 10^y is rational and is equal to that number. If

every rational number is either in one set or in the other we have a process for determining, in regard to every rational number, whether it is greater or less than 10^y . That is to say 10^y is a known irrational number.

We have shown how to define 10^y for any positive value of y , and it is easy to prove from the definition that 10^y obeys the laws of indices for all positive values of y . We define 10^{-y} , for y positive, as $\frac{1}{10^y}$.

The theory here described may be compared with the process suggested in Ch. VI for forming a Table of logarithms by finding, correctly to as many places of decimals as may be wished, the square root, fourth root, and so on, of 10 and of integral powers of 10.

We have seen how, beginning with a positive known index y , rational or irrational, we can define 10^y , and we have also seen how to extend the definition to negative values of y . We have now to show that, if we begin with a positive number x we can define a number y which is such that $10^y = x$. We may suppose x to be greater than 1; for, if x were less than 1, $\frac{1}{x}$ would be greater than 1, and if we could define a number y' so that $10^{y'}$ is equal to $\frac{1}{x}$ then y would be $-y'$. Take any rational number C , we know how to define 10^C , and if it is not equal to x it must be either greater or less. We can sort out all the rational numbers C into two sets, those which make 10^C greater than x and those which make 10^C less than x . There may be one rational number C which is in neither set. There cannot be more than one. If there is one, that one is the required y , and y is rational. If there is not one, then there is an irrational number y which is such that (i) for every rational number a , which is greater than y , $10^a > x$, and (ii) for every rational number b , which is less than y , $10^b < x$. Then, by the definition of 10^y , we have $10^y = x$.

IV. THE EXPONENTIAL LIMIT

It is to be proved that, as h tends to zero, $\left(1 + \frac{h}{x}\right)^{\frac{x}{h}}$ tends to a limit. We begin by taking h to pass through such a sequence of values that $\frac{x}{h}$ is always a positive integer n , and we show that, as n increases, $\left(1 + \frac{1}{n}\right)^n$ increases*, but is always less than 4.

Now
$$\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n$$

if
$$\left(1 + \frac{1}{n+1}\right)^{\frac{n+1}{n}} > 1 + \frac{1}{n},$$

and this is the case if

$$\left(1 + \frac{1}{n+1}\right)^{\frac{n+1}{n}} - 1 > \frac{1}{n},$$

or again if
$$n \left\{ \left(1 + \frac{1}{n+1}\right)^{\frac{n+1}{n}} - 1 \right\} > 1.$$

We put
$$a = \left(1 + \frac{1}{n+1}\right)^{\frac{1}{n}},$$

so that
$$a^n = \left(1 + \frac{1}{n+1}\right),$$

and therefore $(a^n - 1)(n+1) = 1$. Then we have to prove that

$$n(a^{n+1} - 1) > (n+1)(a^n - 1).$$

Now this is true for any value of a greater than 1. In fact, we have

$$a^{n+1} - 1 = (a - 1)(a^n + a^{n-1} + a^{n-2} + \dots + a^2 + a + 1),$$

and

$$a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + \dots + a^2 + a + 1),$$

* Cf. G. Chrystal, *Algebra*, Part II. p. 77 (Edinburgh, 1889).

so that the inequality we have to prove is

$$\begin{aligned} n(a-1)a^n + n(a-1)(a^{n-1} + a^{n-2} + \dots + a + 1) \\ > (n+1)(a-1)(a^{n-1} + a^{n-2} + \dots + a + 1), \end{aligned}$$

or

$$n(a-1)a^n > (a-1)(a^{n-1} + a^{n-2} + \dots + a + 1).$$

On the right-hand side we have n terms, each of which is less than $(a-1)a^n$, a being greater than 1, and therefore the inequality is proved.

Again, we show that, as n increases, $\left(1 - \frac{1}{n}\right)^{-n}$ diminishes.

Now
$$\left(1 - \frac{1}{n}\right)^{-n} > \left(1 - \frac{1}{n+1}\right)^{-(n+1)},$$

if
$$\frac{1}{\left(1 - \frac{1}{n}\right)^n} > \frac{1}{\left(1 - \frac{1}{n+1}\right)^{n+1}},$$

or if
$$\left(1 - \frac{1}{n}\right)^n < \left(1 - \frac{1}{n+1}\right)^{n+1},$$

or if
$$1 - \frac{1}{n} < \left(1 - \frac{1}{n+1}\right)^{\frac{n+1}{n}},$$

and this is the case if

$$1 - \left(1 - \frac{1}{n+1}\right)^{\frac{n+1}{n}} < \frac{1}{n},$$

or again if
$$n \left\{ 1 - \left(1 - \frac{1}{n+1}\right)^{\frac{n+1}{n}} \right\} < 1.$$

We put
$$b = \left(1 - \frac{1}{n+1}\right)^{\frac{1}{n}},$$

so that $b^n = 1 - \frac{1}{n+1}$, and therefore $(1 - b^n)(n+1) = 1$. Then we have to prove that $n(1 - b^{n+1}) < (n+1)(1 - b^n)$.

Now this is true for any value of b less than 1. In fact we have

$$1 - b^{n+1} = (1 - b)(1 + b + b^2 + \dots + b^{n-2} + b^{n-1} + b^n),$$

$$\text{and } 1 - b^n = (1 - b)(1 + b + b^2 + \dots + b^{n-2} + b^{n-1}),$$

so that the inequality we have to prove is

$$\begin{aligned} n(1 - b)(1 + b + b^2 + \dots + b^{n-2} + b^{n-1}) + n(1 - b)b^n \\ < (n + 1)(1 - b)(1 + b + b^2 + \dots + b^{n-2} + b^{n-1}) \end{aligned}$$

$$\text{or } n(1 - b)b^n < (1 - b)(1 + b + b^2 + \dots + b^{n-2} + b^{n-1}).$$

On the right-hand side we have n terms, each of which is greater than $(1 - b)b^n$, b being less than 1, and therefore the inequality is proved.

$$\text{Now, if } n = 2, \quad \left(1 - \frac{1}{n}\right)^{-n} = \left(1 - \frac{1}{2}\right)^{-2} = 4.$$

$$\text{Also } \left(1 + \frac{1}{n}\right)^n \times \left(1 - \frac{1}{n}\right)^n = \left(1 - \frac{1}{n^2}\right)^n,$$

which is less than 1 when n is greater than 1, and therefore

$$\left(1 + \frac{1}{n}\right)^n < \left(1 - \frac{1}{n}\right)^{-n}.$$

$$\text{It follows that } \left(1 + \frac{1}{n}\right)^n < 4.$$

We have therefore proved that, as the integer n increases, $\left(1 + \frac{1}{n}\right)^n$ increases, but remains always less than 4. It therefore tends to a limit (see pp. 186, 187). We call this limit e .

Next we take the case where h tends to zero in such a way that $\frac{h}{x}$ is always positive, but not necessarily an integer. If x is positive, h also is positive, and, as h is diminished towards zero, h passes through all positive numbers, rational or irrational, which are less than some chosen number. Similarly if x is negative. We write z for $\frac{x}{h}$. Then, whatever positive non-

integral value we give to z , there is an integer n such that z lies between n and $n+1$, and, as h tends to zero, n increases indefinitely. Now $\left(1 + \frac{1}{z}\right)^z > \left(1 + \frac{1}{z}\right)^n$ but $< \left(1 + \frac{1}{z}\right)^{n+1}$. Also $1 + \frac{1}{z} > 1 + \frac{1}{n+1}$ but $< 1 + \frac{1}{n}$. Hence $\left(1 + \frac{1}{z}\right)^z$ lies between $\left(1 + \frac{1}{n+1}\right)^n$ and $\left(1 + \frac{1}{n}\right)^{n+1}$, i.e. between $\frac{n+1}{n+2} \left(1 + \frac{1}{n+1}\right)^{n+1}$ and $\frac{n+1}{n} \left(1 + \frac{1}{n}\right)^n$. As the integer n increases indefinitely, both these expressions tend to the same limit e , and therefore $\left(1 + \frac{1}{z}\right)^z$ tends to e as a limit.

Finally, let $\frac{h}{x}$ be negative, and write z for $\frac{h}{x}$. Put $z = -z'$, then z' is positive, and as h tends to zero, z' increases indefinitely. Now

$$\left(1 + \frac{1}{z}\right)^z = \left(1 - \frac{1}{z'}\right)^{-z'} = \left(\frac{z'}{z'-1}\right)^{z'} = \frac{z'}{z'-1} \left(1 + \frac{1}{z'-1}\right)^{z'-1}.$$

As z' increases indefinitely, this expression tends to e as a limit.

It is not relevant to the argument, although it is true, that the limit e in question is the same as that to which the expression

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!}$$

tends as the integer k increases indefinitely. Here $k!$ means the product $2 \times 3 \times 4 \times \dots (k-1) \times k$. This result may be used to compute e approximately, but it is not necessary to know it in order to prove that $\left(1 + \frac{1}{z}\right)^z$ tends to a limit.

A parallel case is presented by the number π . We may prove that the area of a regular polygon of 2^n sides inscribed in a circle tends to a certain limiting area as the integer n increases indefinitely, by showing that as n increases the area increases, and yet that it remains always less than the area of the circumscribed square; and we may call the numerical measure of the limiting area πr^2 when the numerical measure of the radius is r . It is

not relevant to the argument, although it is true, that the number π is also the limit to which the expression

$$4 \left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + (-1)^k \frac{1}{2k+1} \right\},$$

tends as k increases indefinitely, but this result might be used to compute π approximately.

For the calculation of e we use the theorem of § 54, viz. : that, if $f(x)$ vanishes when $x=a$ and when $x=b$, $f'(x)$ vanishes for some value of x between a and b . We use also the result $\frac{de^x}{dx} = e^x$.

We know that $e > 1$. Write R_1 for $e - 1$. Then put

$$f_1(x) = e - e^x - (1-x)R_1.$$

We see that $f_1(x)$ vanishes when $x=0$ and also when $x=1$, and therefore $f_1'(x)$ vanishes for some value of x between 0 and 1. But $f_1'(x) = -e^x + R_1$. Hence R_1 is equal to the value of e^x for some value of x between 0 and 1. We see that $R_1 > 1$, and therefore that $e > 2$.

Put $R_2 = R_1 - 1 = e - 2$,

and $f_2(x) = e - e^x - (1-x)e^x - (1-x)^2 R_2$.

Then $f_2(x) = 0$ when $x=1$ and also when $x=0$, and therefore $f_2'(x)$ vanishes for some value of x between 0 and 1. But

$$f_2'(x) = -e^x + e^x - (1-x)e^x + 2(1-x)R_2 = 2(1-x)(R_2 - \frac{1}{2}e^x).$$

Hence R_2 is equal to the value of $\frac{1}{2}e^x$ for some value of x between 0 and 1.

This value of x is, of course, not the value which made $R_1 = e^x$. We see that $R_2 > \frac{1}{2}$, and, since $e = 2 + R_2$, that $e > 2.5$. Also we see that $R_2 < \frac{1}{2}e$, and therefore that $e < 2 + \frac{1}{2}e$, or $e < 4$.

Put $R_3 = R_2 - \frac{1}{2} = e - 2 - \frac{1}{2}$,

and $f_3(x) = e - e^x - (1-x)e^x - \frac{1}{2}(1-x)^2 e^x - (1-x)^3 R_3$.

Then $f_3'(x) = 0$ when $x = 1$ and also when $x = 0$, and therefore $f_3'(x)$ vanishes for some value of x between 0 and 1. But

$$\begin{aligned} f_3'(x) &= -e^x + e^x - (1-x)e^x + (1-x)e^x - \frac{1}{2}(1-x)^2 e^x + 3(1-x)^2 R_3 \\ &= 3(1-x)^2 \left(R_3 - \frac{1}{2 \times 3} e^x \right). \end{aligned}$$

Hence R_3 is equal to the value of $\frac{1}{6}e^x$ for some values of x between 0 and 1.

We see that $R_3 > \frac{1}{6}$ but $< \frac{1}{6}e$. Hence $e > 2 + \frac{1}{2} + \frac{1}{6}$, or $\frac{8}{3}$, but $e < 2 + \frac{1}{2} + \frac{1}{6}e$, or $\frac{5}{6}e < \frac{5}{2}$, or $e < 3$. Thus e lies between $\frac{8}{3}$ and 3.

Put
$$R_4 = R_3 - \frac{1}{6} = e - 2 - \frac{1}{2} - \frac{1}{6},$$

and

$$f_4'(x) = e - e^x - (1-x)e^x - \frac{1}{2}(1-x)^2 e^x - \frac{1}{6}(1-x)^3 e^x - (1-x)^4 R_4.$$

Then $f_4'(x)$ vanishes when $x = 0$ and also when $x = 1$, and therefore $f_4'(x)$ vanishes for some value of x between 0 and 1.

But
$$f_4'(x) = -\frac{1}{6}(1-x)^3 e^x + 4(1-x)^3 R_4,$$

for all the remaining terms cancel, or we have

$$f_4'(x) = 4(1-x)^3 \left(R_4 - \frac{1}{2 \times 3 \times 4} e^x \right).$$

Hence $R_4 = \frac{1}{24}e^x$ for some value of x between 0 and 1. We

see that $R_4 > \frac{1}{24}$ but $< \frac{1}{24}e$. Hence $e > 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$, but

$$e < 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}e \quad \text{or} \quad \frac{23}{24}e < \frac{8}{3}.$$

Thus e lies between $\frac{65}{24}$ and $\frac{64}{23}$.

We can proceed in this way, getting closer and closer approximations to e . Put in general

$$R_k = e - 2 - \frac{1}{2} - \frac{1}{2 \times 3} - \frac{1}{2 \times 3 \times 4} - \dots - \frac{1}{2 \times 3 \times 4 \times \dots (k-1)},$$

and

$$\begin{aligned} f_k(x) = e - e^x - (1-x)e^x - \frac{(1-x)^2}{2}e^x - \frac{(1-x)^3}{2 \times 3}e^x \\ - \dots - \frac{(1-x)^{k-1}}{2 \times 3 \dots (k-1)}e^x - (1-x)^k R_k. \end{aligned}$$

Then $f_k(x)$ vanishes when $x=0$ and also when $x=1$, and therefore $f'_k(x)$ vanishes for some value of x between 0 and 1. But, just as before, we find

$$\begin{aligned} f'_k(x) &= -\frac{(1-x)^{k-1}}{2 \times 3 \times \dots (k-1)}e^x + k(1-x)^{k-1}R_k \\ &= k(1-x)^{k-1} \left\{ R_k - \frac{e^x}{2 \times 3 \times \dots (k-1)k} \right\}, \end{aligned}$$

and therefore $R_k = \frac{e^x}{2 \times 3 \times \dots k}$ for some value of x between 0 and 1, or

$$e = 2 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \dots + \frac{1}{2 \times 3 \times \dots (k-1)} + \frac{e^x}{2 \times 3 \times \dots k},$$

for some value of x between 0 and 1. Thus e lies between

$$2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{2 \times 3 \times \dots (k-1)} + \frac{1}{2 \times 3 \times \dots k},$$

and
$$2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{2 \cdot 3 \dots (k-1)} + \frac{3}{2 \times 3 \times \dots k},$$

for $e^x < 3$ when $x < 1$.

We see that we can calculate e to any desired degree of accuracy.

V. THE MENSURATION OF THE CIRCLE AND THE RADIAN MEASURE OF ANGLES.

Let the radius of a circle be r units of length. Let squares be circumscribed about and inscribed in the circle. Fig. 80 shows at once that their areas are $4r^2$ and $2r^2$ units of area. The straight lines drawn through the centre to bisect the sides of the inscribed square, meet the circumference in 4 points, which, together with the 4 corners of the square, make up the 8 corners of an inscribed regular octagon. Fig. 80 shows that the area of

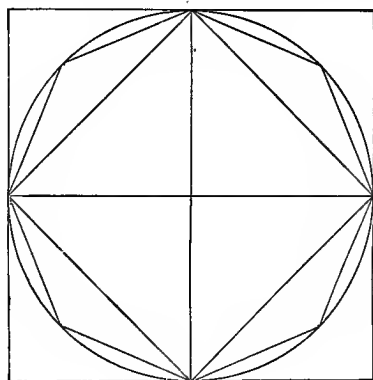


Fig. 80.

the octagon is greater than that of the inscribed square. In like manner, by bisecting the sides of the octagon, we may find the 16 corners of a regular inscribed polygon of 16 sides, and the area of this polygon is greater than that of the octagon. We may proceed to inscribe regular polygons of any number of sides, the number being a power of 2, and we see that, as the number of sides increases, the area of the polygon always increases. But, as no point in any inscribed polygon can be outside the circle, the areas of all the inscribed polygons are less than the area of

the circumscribed square. Hence the number of units of area in the area of the regular polygon of 2^n sides inscribed in the circle increases as n increases, but never becomes so great as $4r^2$. It therefore tends to a limit (see pp. 186, 187). This limit is defined to be the number of units of area in the area of the circle.

Let regular polygons of 2^n sides be inscribed in two circles whose radii are r_1, r_2 units of length. The two polygons are similar figures and their areas are proportional to the areas of the squares described upon corresponding lines in the two figures, and therefore the measures of their areas are proportional to the numbers r_1^2, r_2^2 , and the areas are expressible as $f_n r_1^2$ and $f_n r_2^2$ units of area, where f_n is a number which depends upon n , but not upon r_1 or r_2 . As n increases, the number f_n tends to a limit. This limit is called π . The area of the circle, of radius r units of length, is πr^2 units of area.

Let the length of the perimeter of the inscribed regular polygon of 2^n sides be p units of length, and let the length of the perpendicular drawn from the centre to any side of this polygon be q units of length. The area of the polygon is $\frac{1}{2}pq$ units of area. Now, as n increases, q tends to a limit, which is r . Hence p tends to a limit which is $2\pi r$. The limit to which the length of the perimeter of the polygon tends is defined to be the length of the circumference of the circle. We learn that it is $2\pi r$ units of length.

When we inscribe in the circle a regular polygon of 2^n sides, we divide the part of the plane which lies within the circle into 2^n congruent sectors, and the circumference into 2^n congruent arcs, and the angles subtended by these arcs at the centre are all equal. The area of each sector is $\frac{\pi r^2}{2^n}$ units of area, the length of each arc is $\frac{2\pi r}{2^n}$ units of length, and the measure of each angle is $\frac{4}{2^n}$ right angles.

Now let AB be any arc of the circle, and let A be one vertex of an inscribed regular polygon of 2^n sides. If, for any value of n , B is another vertex, there may or may not be other vertices on the arc AB . We can include both these cases in the same statement by taking the number of intermediate vertices to be $m - 1$, where m may be 1 or may be greater than 1. Then the length of the arc AB is $\frac{m}{2^n} 2\pi r$ units of length, the magnitude of the angle which it subtends at the centre is $4 \frac{m}{2^n}$ right angles, and the area of the corresponding sector is $\frac{m}{2^n} \pi r^2$ units of area.

If, however, B is not a vertex for any value of n , we may suppose that, when the polygon has 2^n sides, there are m vertices on the arc AB between A and B (A not counted as one). Then the magnitude of the angle subtended by the arc AB at the centre lies between $4 \frac{m}{2^n}$ and $4 \frac{m+1}{2^n}$ right angles, and the end B of the arc AB lies between two points P and Q which are at the ends of arcs AP and AQ of lengths $\frac{m}{2^n} 2\pi r$ and $\frac{m+1}{2^n} 2\pi r$ units of length, also the areas of the sectors standing on the arcs AP and AQ are $\frac{m}{2^n} \pi r^2$ and $\frac{m+1}{2^n} \pi r^2$ units of area.

We write x_n and y_n for the numbers $\frac{m}{2^n}$ and $\frac{m+1}{2^n}$, and consider how the numbers x_n and y_n are altered when n is increased. When n is changed into $n+1$, the number of sides of the inscribed polygon is doubled, and new vertices are introduced. These new vertices lie on the bisectors of the angles which are subtended at the centre by the sides of the polygon of 2^n sides. We name these angles the first, the second, and so on, beginning with that angle of which one side passes through A . If the bisector of the $(m+1)$ th angle does not cut the circumference at a point on the arc AB , m is changed to $2m$; if it does, m is

changed to $2m + 1$. When n is changed to $n + 1$ and m to $2m$, $x_{n+1} = x_n$, but

$$y_{n+1} = \frac{2m + 1}{2^{n+1}} < \frac{m + 1}{2^n}, \text{ or } y_{n+1} < y_n.$$

When n is changed to $n + 1$ and m to $2m + 1$,

$$x_{n+1} = \frac{2m + 1}{2^{n+1}} > \frac{m}{2^n}, \text{ or } x_{n+1} > x_n,$$

but

$$y_{n+1} = \frac{2m + 2}{2^{n+1}} = \frac{m + 1}{2^n}, \text{ or } y_{n+1} = y_n.$$

If we keep on doubling the number of sides of the polygon, at every step one of the two numbers x_n , y_n is changed and the other is not. When x_n is changed it is increased, when y_n is changed it is diminished. But, since the magnitude of the angle subtended by AB at the centre always lies between $4x_n$ and $4y_n$ right angles, all the numbers x_n are less than any one of the numbers y_n . Further $y_n - x_n = \frac{1}{2^n}$ and this can be made as small as we please by increasing n sufficiently. It follows that, as n increases, x_n and y_n tend to a common limit (see p. 186).

Let us denote this limit by $\frac{a}{2\pi}$. Then the angle which AB subtends at the centre is $\frac{2a}{\pi}$ right angles, the lengths of the arcs AP and AQ tend to one and the same limiting length, which is $a\pi$ units of length, and the areas of the sectors standing on AP and AQ tend to one and the same limiting area, which is $\frac{1}{2}a\pi^2$ units of area. We define this limiting length and limiting area to be the length of the arc AB and the area of the sector standing on this arc.

It appears that, whether B is a vertex or not, there is a number a which is such that (i) the angle which the arc AB subtends at the centre is $\frac{2a}{\pi}$ right angles, (ii) the length of the

arc AB is ar units of length, (iii) the area of the sector standing on this arc is $\frac{1}{2}ar^2$ units of area. The number a measures the angle in terms of a certain angle chosen as unit. This unit angle is the "radian." Its magnitude is $\frac{2}{\pi}$ right angles.

VI. TRIGONOMETRIC LIMITS.

If the lengths of an arc and chord of a circle are l and χ units of length, then, as the ends of the chord approach to coincidence, $\frac{l}{\chi}$ tends to 1 as a limit.

Let the chord PQ be bisected in N, let the straight line joining the centre C to N meet the circle in A, as shown in Fig. 81, and let the tangent PT meet the straight line CA in T. Let the angle ACP be a radians, and the radius of the circle r units of length. Then the lengths of the chord PQ and arc PQ are $2r \sin a$ and $2ra$ units of length. We have therefore to prove that, as a tends to zero, $\frac{a}{\sin a}$ tends to 1 as a limit.

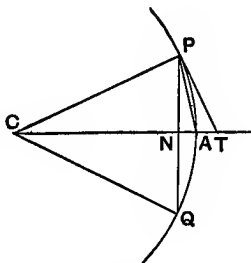


Fig. 81.

The lengths of PN, the arc PA, and PT are $r \sin a$, ra , and $r \tan a$ units of length, and the areas of the triangle PCA, the sector PCA, and the triangle PCT are $\frac{1}{2}r^2 \sin a$, $\frac{1}{2}r^2 a$, and $\frac{1}{2}r^2 \tan a$ units of area*. Hence

$$\sin a < a < \tan a,$$

or we have

$$1 < \frac{a}{\sin a} < \sec a.$$

* Cf. E. W. Hobson, *Trigonometry*, Ch. VIII

When α tends to zero $\sec \alpha$ tends to 1 as a limit, and therefore $\frac{\alpha}{\sin \alpha}$, which lies between 1 and $\sec \alpha$, tends to 1 as a limit.

We may deduce the result that when α tends to zero $\frac{1 - \cos \alpha}{\alpha}$ tends to zero as a limit. We have

$$\sin^2 \alpha = (1 - \cos \alpha)(1 + \cos \alpha),$$

and therefore

$$\frac{1 - \cos \alpha}{\alpha} = \frac{\sin \alpha}{\alpha} \frac{1}{1 + \cos \alpha} \sin \alpha.$$

Now when α tends to zero, $\cos \alpha$ tends to 1 as a limit, and $1 + \cos \alpha$ tends to 2 as a limit, and therefore the limits of the three factors on the right-hand side are 1, $\frac{1}{2}$, 0. This proves the result stated.

By means of these two results and the addition equation for the sine,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

we may differentiate $\sin x$. We have

$$\sin(x + h) = \sin x \cos h + \cos x \sin h,$$

and therefore

$$\begin{aligned} \frac{\sin(x + h) - \sin x}{h} &= \frac{\sin h \cos x - (1 - \cos h) \sin x}{h} \\ &= \cos x \frac{\sin h}{h} - \sin x \frac{1 - \cos h}{h}. \end{aligned}$$

The limit to which the right-hand member tends as h tends to zero is $\cos x$.

In like manner we have the addition equation for the cosine,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

and therefore

$$\begin{aligned} \frac{\cos(x + h) - \cos x}{h} &= - \frac{\sin x \sin h + \cos x (1 - \cos h)}{h} \\ &= - \sin x \frac{\sin h}{h} - \cos x \frac{1 - \cos h}{h}, \end{aligned}$$

and, as h tends to zero, the right-hand member tends to $-\sin x$ as a limit.

This method of proof appears much shorter than that given in Ch. VII., but, to render it complete, we should require to prove the formulæ for $\sin(A + B)$ and $\cos(A + B)$ for all values of A and B , positive and negative. This is, of course, done in books on Trigonometry. The method used in Ch. VII. shows that $\sin x$ and $\cos x$ can be differentiated without proving these formulæ.

VII. MECHANICAL UNITS.

In the Chapters of this book, containing applications of the Calculus to Mechanics, the units employed are those of the so-called "British Engineers' System." In this system of units the fundamental quantities are force, length, time. Mass is a derived quantity. The unit of force is the force required to support in London a body which weighs 1 lb. in a common balance. This is the "force of 1 lb." The units of length and time are 1 foot and 1 second. The unit of mass is adjusted so that the force of 1 lb. acting upon a body whose mass is 1 unit of mass may produce in it 1 unit of acceleration; it is therefore the mass of a body which weighs 32.2 lbs. in a common balance.

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